Geometric Group Theory: mini-projects for the Broadening Course Assessment

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1 Accessibility for groups

The goal of the project is to overview the example of Dunwoody showing that the following theorem from the Lecture Notes does not generalize to finitely generated groups.

Theorem 1.1. *(Dunwoody) Let* Γ *be a finitely presented group. Then* Γ *can be written as* $\Gamma = \pi_1(G, Y, a_0)$ *where* (G, Y) *is a finite graph of groups such that all edge groups are finite and all vertex groups do not split over finite groups.*

The first example appeared in the paper of M.J. Dunwoody, *An inaccessible group*, Geometric group theory, Vol. 1 (Sussex, 1991), 75-78, London Math. Soc. Lecture Note Ser., 181, Cambridge Univ. Press, Cambridge, 1993.

Since then other examples torsion-free have appeared, they can eventually also be studied.

The project could have as a starting point the nice construction provided in the paper above, and may further develop following any of the ideas explained in Section 20.6 of Druţu-Kapovich, "Geometric Group Theory".

2 Real trees. The universal real tree

Simplicial trees have generalisations called *real trees*: one needs to imagine trees, in which distances between two branching points are not necessarily integers, and the number of directions in a branching point is not necessarily countable, but may be of larger cardinalities as well.

Among the real trees of power continuum there exist some universal ones: any other tree of power continuum has an isometric embedding into such a universal tree. Their explicit construction and universality property are described in the paper of A. Dyubina and I. Polterovich, *Explicit constructions of universal R-trees and asymptotic geometry of hyperbolic spaces*, Bull. London Math. Soc. 33 (2001), no. 6, 727?734.

The project could investigate real trees in general, and this particular construction.

3 A proof of Stallings' Theorem

The goal of this mini-project is to provide a sketch of Stallings' theorem:

Theorem 3.1. *If G is a finitely generated group with infinitely many ends then G is admits a non-trivial decomposition as a graph of groups with finite edge groups.*

The reference is Chapter 21 in Druţu-Kapovich, "Geometric Group Theory".

4 Boundaries of hyperbolic groups

This project is to investigate the notion of ideal boundary of a hyperbolic space and of a hyperbolic group, and the properties of the boundary extensions of the quasi-isometries and of the quasi-actions.

The reference is M. Gromov's book, "Hyperbolic groups", and Sections 11.11 to 11.15 in Druţu-Kapovich, "Geometric Group Theory".

5 Quasi-isometric rigidity of fundamental groups of non-geometric Haken manifolds with zero Euler characteristic

A remarkable theorem of Kapovich-Leeb states: if a finitely generated group with a word metric is quasi-isometric to the fundamental groups of a non-geometric Haken manifold with zero Euler characteristic, then, up to taking a quotient by a finite normal subgroup and taking a finite index subgroup, the group is isomorphic to the fundamental group of a Haken manifold of the same kind. This has been done by Kapovich-Leeb in the non-geometric case, in the geometric case it follows from work of several authors. The project could focus on the work of Kapovich-Leeb, or on the contrary on the geometric case, pick some (or all) of the eight geometries and explain how a similar result follows from the literature.

References are:

M. Kapovich, B. Leeb, *Quasi-isometries preserve the geometric decomposition of Haken manifolds*, Invent. Math. 128 (1997), no. 2, 393–416.

Chapter 25 of Druţu-Kapovich and references therein.

6 Rapid Decay Property for free and hyperbolic groups

The property of Rapid Decay can be seen as a non-commutative generalization of the property of density of the algebra of smooth functions on a torus $\mathbb{T}^n$ inside the algebra of continuous functions on $\mathbb{T}^n$; it requires the existence of an analogue of the Schwartz space inside the reduced C^*–algebra of a group. The significance of property RD is emphasized by a consequence of it which goes back to Swan and Karoubi stating that if the algebra of functions with Rapid Decay $H^\infty(G)$ is contained in $C_r^*(G)$ then the inclusion induces an isomorphism of K–groups $K_i\left(H^\infty(G)\right)$ with $K_i\left(C_r^*(G)\right)$, for $i = 0, 1$. This is a main ingredient in the proof due to Connes and Moscovici of the Novikov conjecture for Gromov hyperbolic groups.

This project is supposed to investigate property RD as formally defined by Jolissaint, (some of) its equivalent definitions, immediate properties, and the proof that free groups have property RD, possibly also the proof that Gromov hyperbolic groups have property RD. Both for the free group and Gromov hyperbolic groups the recommended reference is the book of Alain Connes, section III.5.α.

References are:

P. Jolissaint, *Rapidly decreasing functions in reduced C*-algebras of groups*, Trans. Amer. Math. Soc. 317 (1990), no. 1, 167–196.

A. Connes, *Noncommutative Geometry*, Academic Press (1994).

7 Compression of embeddings of graphs. Compression of embeddings of trees and hyperbolic groups into Hilbert spaces

In the study of finite graphs one often uses methods in which graphs are embedded into various metric spaces with "good" properties. From this viewpoint, the notions of dilatation and compression are particularly significant (see the corresponding sections in the survey of Sh. Hoory, N. Linial, A. Wigderson for definitions and properties).

These notions have appropriate generalizations for infinite graphs. This project is supposed to investigate the very short proof of Linial and Saks of a theorem of Bourgain on embeddings of simplicial trees in Hilbert spaces, possibly also its generalization to hyperbolic groups due to N. Brodskiy and D. Sonkin.

References are:

Sh. Hoory, N. Linial, A. Wigderson, *Expander graphs and their applications*, Bull. Amer. Math. Soc. 43 (2006), 439–561.

N. Linial, M. Saks, *The Euclidean distortion of complete binary trees*, Discrete Comput. Geom. 29 (2003), 19–21.

N. Brodskiy, D. Sonkin, *Compression of uniform embeddings into Hilbert space*, preprint, 2005, arXiv:math.GR/0509108.

8 Any other topic related to the course

Any topic related to the course that you would be interested in investigating further, either in the book Druţu-Kapovich, or in another book, or in a published paper. Let me know, and (eventually with some adjustments) well formulate a project.