

C2.6 Introduction to Schemes

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HT 2020

Feedback and corrections are welcome!

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EXERCISE SHEET 4

- i) $F, G \subseteq H$ subsheaves then $F = G \iff F_x = G_x \quad \forall x \in X$
- ii) $F \xrightarrow{\varphi} G$ in $\text{Ab}(X) \implies \text{Ker } \varphi : U \mapsto \text{Ker}(\varphi_U)$ is a sheaf (whereas for $\text{Coker } \varphi$ and $\text{Im } \varphi$ we must sheafify)
- iii) Also show $(\text{Ker } \varphi)_x = \text{Ker}(\varphi_x)$ and $(\text{Im } \varphi)_x = \text{Im}(\varphi_x)$ (recall the definition $\text{Im } \varphi = \text{Ker}(G \rightarrow \text{Coker } \varphi)$)
- Show φ injective $\iff \varphi_x$ injective $\forall x$, φ surjective $\iff \varphi_x$ surjective $\forall x$
- Deduce that $F \rightarrow G \rightarrow H$ in $\text{Ab}(X)$ exact $\iff F_x \rightarrow G_x \rightarrow H_x$ exact $\forall x$
- iv) $F \xrightarrow{\varphi} G$ with φ_x surjective $\implies \forall s \in G(U), x \in U, \exists$ open $v \in V \subseteq U, \exists t \in F(V)$ with $\varphi(t) = s|_V$
- v) "surjectivity means local liftability" $F \xrightarrow{\varphi} G$ surj $\iff \forall s \in G(U), \exists$ open cover $U = \cup U_i, \exists t_i \in F(U_i)$ with $F(t_i) = s|_{U_i}$
- vi) $X = \mathbb{C} \setminus \{ \frac{1}{n} : n \geq 1 \in \mathbb{N} \}$ $\mathcal{O}_X(U) := \{ \text{holomorphic functions } U \rightarrow \mathbb{C} \}$ (holomorphic = complex-differentiable)
Euclidean topology $\mathcal{O}_X^*(U) := \{ \text{nowhere zero holomorphic functions } U \rightarrow \mathbb{C} \}$
 Show $\exp : \mathcal{O}_X \rightarrow \mathcal{O}_X^*$ is surjective (for $f \in \mathcal{O}_X(U), \exp(f) \in \mathcal{O}_X^*(U)$ is the complex exponential)
 but $\exp_U : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X^*(U)$ not surjective no matter how small the open $U \in \mathcal{O}_X$.
- vii) $F \xrightarrow{\varphi} G$ hom of \mathcal{O}_X -mods, G finite type then: φ_x surj. $\implies \varphi_U : F|_U \rightarrow G|_U$ surj. (In notes 6.3 we used this: we had $\mathcal{O}_{X,x}^{\oplus n} \cong F_x$, and assuming F finite type we claimed that $\mathcal{O}_U^{\oplus n} \rightarrow F|_U$ surjective on some open $x \in U$.)
- viii) $F \xrightarrow{\varphi} G$ hom of \mathcal{O}_X -mods, G coherent then: φ_x inj. $\implies \varphi_U : F|_U \rightarrow G|_U$ inj. (Hint First check $\text{Ker } \varphi$ finite type (doesn't use φ_x inj) by considering $\text{Ker}(\mathcal{O}_U^{\oplus n} \rightarrow F|_U \xrightarrow{\varphi_U} G|_U)$)

2) Motivation: why is Nakayama's Lemma useful in geometry?

"Transferring information from pointwise to infinitesimal to local": ($\cong M_p/p \cdot M_p$)

Recall Nakayama's Lemma: (there are many versions of this, the proofs are very similar)

R ring, $p \in \text{Spec } R$, M f.g. R -mod. If $n_1, \dots, n_d \in M$ is a basis for the $K(p)$ -vector space $M_p \otimes_{R_p} K(p)$ then n_1, \dots, n_d generate the R_p -module M_p for some $f \in R \setminus p$ (indeed it is a minimal generating set)

(when R is a local ring with max ideal m , this becomes: $M/mM = \langle n_1, \dots, n_d \rangle \implies M = \langle n_1, \dots, n_d \rangle$)

i) (X, \mathcal{O}_X) scheme, $F \in \text{Qcoh}(X)$ finite type then call $F(x) = F_x \otimes_{\mathcal{O}_{X,x}} K(x)$ the fiber

Given $s_1, \dots, s_n \in F(U)$ on open $x \in U$, if $(s_1)_x, \dots, (s_n)_x$ generate the fiber then possibly after shrinking U , show the s_1, \dots, s_n also generate $F|_U$.

Deduce: • if $F(x) = 0$ then $F|_U = 0$ some open $x \in U$.

• $x \mapsto \dim F(x)$ is upper-semicontinuous, i.e. $\{ \dim < d \} \subseteq X$ is open (since integer-valued can also take $\leq d$.)

Algebra fact in R -mods: $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ exact, M_2 flat (e.g. free) then

M_3 flat $\iff M_1/I M_1 \rightarrow M_2/I M_2$ injective \forall ideal $I \subseteq R$.

ii) Let $F \in \mathcal{O}_X$ -mods be locally finitely presented. Prove: $F \in \text{Vect}(X) \iff F$ flat \mathcal{O}_X -mod

(Hints. Rewrite the algebra fact in case R local ring, check it's enough to consider $I =$ the max ideal. The key is to reach an exact sequence of type $0 \rightarrow N_x/m_x N_x \rightarrow K(x) \otimes^n \rightarrow F_x/m_x F_x \rightarrow 0$ & use (i))

3) Motivation: $\text{Vect}(\text{Spec } R) \leftrightarrow \tilde{M}$ for f.g. projective R -mods M

$X = \text{Spec } R, M \text{ } R\text{-mod.}$

Consider the conditions

[You may need: projective R -mod \Leftrightarrow direct summand of free R -mod
also: P projective \Leftrightarrow every exact $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ splits]

① $\tilde{M} \in \text{Vect}(X)$ i.e. locally free of finite rank, i.e. \exists cover $X = \cup D_{f_i}$ with M_{f_i} f.g. free R_{f_i} -mod

① M finitely presented and flat

② M finitely presented and M_m free R_m -mod \forall max ideal m (can also use all prime ideals)

③ M is the direct summand of a finite rank free module

④ M f.g. projective

i) Prove ③ \Leftrightarrow ① Hints for \Rightarrow use ex. 2(ii), for \Leftarrow compare the proof in Sec. 3.1 of notes, use tricks from Sec. 3.0, and use fact $0 \rightarrow K \rightarrow M_1 \rightarrow M_2 \rightarrow 0$ exact $\Rightarrow K$ f.g. M_1 f.g., M_2 finitely presented } $\Rightarrow K$ f.g.

ii) Prove ① & ② \Leftrightarrow ② and ④ \Leftrightarrow ③ \Rightarrow ①

iii) Finally prove ① \Rightarrow ④ (Hint use fact about localization: $S^{-1}\text{Hom}_R(M, N) \cong \text{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ for M finitely presented & use Sec. 3.0 of notes)

4) i) $X = \text{Spec } R, M \text{ } R\text{-mod}$

• Show that $\mathcal{L} = \tilde{M}$ is line bundle $\Leftrightarrow \forall p \in X, \exists f \in R \setminus p: M_f \cong R_f$

• $\mathcal{L} = \tilde{M}$ line bundle $\Leftrightarrow M$ f.g. projective R -mod with $\dim_{K(p)} M \otimes K(p) = 1 \quad \forall p \in X$.

Deduce that every line bundle on $\mathbb{A}_k^1 = \text{Spec } k[t]$ is trivial (k field) \leftarrow Hint structure theorem for f.g. mods over PID

\leftarrow any scheme \leftarrow in terms of transition functions of F

ii) Let $F \in \text{Vect}(X)$. Describe the transition function of the dual $F^\vee := \text{Hom}_{\mathcal{O}_X}(F, \mathcal{O}_X)$.

Deduce that for a line bundle \mathcal{L} , the transition function of \mathcal{L}^\vee is the inverse of that of \mathcal{L}

and $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee = \mathcal{L} \otimes_{\mathcal{O}_X} \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \xrightarrow{\cong} \mathcal{O}_X$ via the natural evaluation map.

iii) Let M, N be R -mods. Suppose $M \otimes_R N \xrightarrow{\varphi} R$. Pick $m_i \in M, n_i \in N$ with $\varphi(\sum_{i=1}^d m_i \otimes n_i) = 1$. Check that $M \rightarrow M, m \mapsto \sum \varphi(m \otimes n_i) m_i$ is an isomorphism which factorizes as $M \rightarrow R^d \rightarrow M$, and deduce that M is a summand of R^d .

iv) $\mathcal{L} \in \mathcal{O}_X\text{-Mods}$ is line bundle $\Leftrightarrow \exists F \in \mathcal{O}_X\text{-Mod}: F \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$ (Def: \mathcal{L} being invertible sheaf)
(Hint combine (iii) with ex. 3(3))

5) FACT every line bundle on \mathbb{A}_k^n is trivial

i) Calculate $\text{Pic}(\mathbb{P}^n) = \{\text{isomorphism classes of line bundles on } \mathbb{P}^n\}$ with group operation $\cdot \otimes_{\mathcal{O}_{\mathbb{P}^n}}$.
Indeed show it is $\cong \mathbb{Z}$, generated by $\mathcal{O}(1)$ (defined in the notes)

ii) Compute $\Gamma(\mathbb{P}^n, \mathcal{O}(d))$ for $d \in \mathbb{Z}$ ($\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$)

iii) Let p be the point $(x) \in \text{Spec } k[x] = A_0$ in $\mathbb{P}_k^1 = A_0 \cup A_1$. Show that $\mathcal{O}(-1) \cong$ ideal sheaf of $\{p\}$.
Let Z be the closed subscheme $\text{Spec}(k[x]/x^d) \subseteq A_0 \subseteq \mathbb{P}_k^1$. Show $\mathcal{O}(-d) \cong$ ideal sheaf of Z
what is the ideal sheaf of d closed points $\{p_1, \dots, p_d\} \subseteq \mathbb{P}^1$? \leftarrow see Sec. 10 notes

iv) OPTIONAL If two graded R -mods M, N over graded ring R satisfy $M_n \cong N_n \quad \forall n \gg d \Rightarrow \tilde{M} = \tilde{N}$

6) i) $\mathcal{C} = \text{abelian cat.}$ Show that if every object $M \in \mathcal{C}$ has an injective morph $M \rightarrow I$ into an injective object, then every object M admits an injective resolution. (We say \mathcal{C} has "enough injectives")

FACT Cat. Ab of abelian groups has enough injectives

ii) $F \in \text{Ab}(X)$. Pick $I_x \in \text{Ab}$ s.t. $F_x \rightarrow I_x$ injective morph and I_x injective object in Ab

Show that $I := \prod_{x \in X} (\mathcal{O}_x)_* I_x \in \text{Ab}(X)$ is an injective object admitting an inj. morph $F \rightarrow I$
 \leftarrow inclusion map $\varphi_x: \{x\} \hookrightarrow X$ of a point. (hence $\text{Ab}(X)$ has enough injectives)