C2.6 Introduction to Schemes

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Feedback and corrections are welcome!

EXERCISE SHEET 3

← Mostly topology, but useful

1) i) (Warm-up lemma) X topological space, check that $\forall$ top. subspace $Y \subseteq X$:
Y irreducible $\Rightarrow$ Y connected
Y irreducible $\Rightarrow$ $\overline{Y}$ irreducible
Y irreducible component $\Rightarrow$ Y closed and connected ← recall: irred. component means irreducible and maximal w.r.t. $\subseteq$

ii) Suppose X has finitely many irreducible components X_i.
Say "X_k can be reached from X_ℓ" if $X_k \cap X_{i_1} \neq \emptyset$, $X_{i_1} \cap X_{i_2} \neq \emptyset, \ldots, X_{i_n} \cap X_\ell \neq \emptyset$ some X_{i_r}.
Prove that X is connected $\Leftrightarrow$ any irred. component can be reached from any other.

iii) A topological space is Noetherian if it satisfies the descending chain condition for closed sets: $C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots \Rightarrow C_N = C_{N+1} = \cdots$ some N.
Prove: a Noeth. top. space has finitely many irreducible components, each containing an open dense set $\neq \emptyset$

iv) R Noeth. ring $\Rightarrow$ Spec R Noeth. top. space. ← (so for a Noetherian scheme every affine open is Noeth. top. space)
Check converse fails for $k[x_1,x_2,x_3,\ldots]/(x_1,x_2^2,x_3^3,\ldots)$.

v) X Noeth. top. space $\Leftrightarrow$ every top. subspace of X is quasi-compact ← so for a Noeth scheme X all subspaces are quasi-compact, not just X.

vi) X Noeth. scheme $\Rightarrow$ X Noeth. top. space

2) i) Check $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ is a variety (k alg. closed field)
(Recall: variety = scheme which is integral, separated, finite type over Spec k.
Hint: You may assume as known that being "f.g. k-alg." is affine-local: notes Sec. 3.2)

ii) Show that the open subscheme $\mathbb{A}^2_k \setminus \{0\}$ is a variety which is not affine

iii) A variety which is affine (Spec(ring)) is an affine variety, i.e. $\cong$ integral closed subscheme of $\mathbb{A}^n_k$ (some n)

iv) $(X, \mathcal{O}_X)$ variety $\Rightarrow$ X Noetherian scheme

v) Glue two copies of $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ along the basic open set $\mathbb{A}^1_k \setminus 0 = D_x = \operatorname{Spec} k[x,x^{-1}]$ by the isomorphism $\operatorname{Spec} k[s,s^{-1}] \xrightarrow{\cong} \operatorname{Spec} k[t,t^{-1}]$ given by $s \mapsto t$.
Show that the glued scheme is not separated. ← (compare notes Sec. 5.3)

vi) OPTIONAL EXERCISE $(X, \mathcal{O}_X)$ variety, $Z \subseteq X$ irreducible subspace ← (Rmk irreducibility is not vital if allow varieties to be reducible.)
In notes Sec. 5.5 you find the definition of what it means for Z to be locally closed $\subseteq$ scheme X and how we construct a canonical induced reduced scheme structure $\mathcal{O}_Z$.

- Prove $\boxed{Z \text{ locally closed} \Rightarrow (Z, \mathcal{O}_Z) \text{ variety}}$ ← (Hint 2(iv), 1(vi), 1(v) may help)
- (harder) if you define $\mathcal{O}_Z$ as suggested in Sec. 5.5 for $Z \subseteq X$ irreducible subspace, prove that $(Z, \mathcal{O}_Z)$ variety $\Rightarrow$ $Z \subseteq X$ locally closed

Suggestion first reduce to affine case $Z = \operatorname{Spec} S$, $X = \operatorname{Spec} R$ by picking Spec $R \subseteq X$ of type open∩closed
Now want an open in Z s.t. generating global sections (over k) come from sections on open $\subseteq X$.
At the end, you may need to check $\operatorname{Spec} S \cap \underbrace{\operatorname{Spec} R_f}_{\{x \in X : f(x) \neq 0 \in k(x)\}} = \operatorname{Spec} S_f$ ← ($S_f = S \otimes_R R_f$ via $\varphi^\# : R \to S$)
where $A = \operatorname{Spec} S \xrightarrow{\subseteq} \operatorname{Spec} R = B$

3) $f: X \longrightarrow B$ morph of schemes

i) f is called an immersion (or locally closed immersion) if $f: X \xrightarrow{\text{closed immersion}} U \xrightarrow{\text{open immersion}} B$

Show that an immersion is a closed immersion $\Longleftrightarrow$ $f(X) \subseteq B$ closed set

(Hint. For $\Leftarrow$: glue the ideal sheaf of $X \xrightarrow{\varphi} U$ with $\mathcal{O}_X|_{X\setminus\varphi(X)}$, check quasi-coherence)

ii) Show $\Delta_{X/B} \subseteq X \times_B X$ is closed if B, X affine $\longleftarrow$ (notation of notes Sec 5.3

iii) Show $\Delta_{X/B}$ immersion (Hence: f separated $\Leftrightarrow \Delta_{X/B}$ closed imm. $\Leftrightarrow \Delta_{X/B}$ closed set)

iv) Call $U, V \subseteq X$ "nice" if $U, V, U\cap V$ affine opens and $\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V) \xrightarrow{\text{surj.}} \mathcal{O}_X(U \cap V)$

- f separated $\Longrightarrow$ ($\forall$ affine open $U, V \subseteq X$ with $f(U), f(V) \subseteq$ affine open in $B \Rightarrow U, V$ nice)
- ($\exists$ open cover $X = \bigcup U_i$ s.t. $\forall x, y \in X$ with $f(x) = f(y)$ $\exists$ nice U_i, U_j with $x \in U_i$, $y \in U_j$ and $f(U_i), f(U_j) \subseteq$ affine open of B) $\Longrightarrow f$ separated

$\Longrightarrow$ For $B = \operatorname{Spec} k$: ($\exists$ open cover $X = \bigcup U_i$, all U_i, U_j nice) $\Rightarrow$ (f separated) $\Rightarrow$ (all affine opens U, V are nice)

v) Show $\mathbb{P}^n_k$ is separated by using (iv) (k any field). Deduce that $\mathbb{P}^n_k$ is a variety.

Show any projective variety ($\leftarrow$ integral closed subscheme of $\mathbb{P}^n_k$) and quasi-projective variety ($\leftarrow$ irreducible open subsch. of a projective variety) are varieties

4) i) Fact $\mathbb{P}^n_k$ is complete (i.e. proper$/k$)

In notes, we showed $\mathbb{A}^1$ is not complete because $\mathbb{A}^1 \times \mathbb{A}^1 \supseteq \mathbb{V}(xy-1) \longrightarrow \mathbb{A}^1$ fails the universally closed condition. Why is this not a problem for $\mathbb{P}^1$ if consider $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$?

ii) $C \subseteq X$ closed subsc., X complete $\Longrightarrow C$ complete $\longleftarrow$ (compare in topology: closed $\subseteq$ compact is compact

So Fact $\Longrightarrow$ also all projective varieties are complete.

iii) $f: X \longrightarrow Y$, X universally closed, Y separated $\Longrightarrow \operatorname{Im}(f) \subseteq Y$ closed & universally closed $\longleftarrow$ $\boxed{\text{compare topology: compact} \xrightarrow{\text{cts}} \text{Hausdorff then image is closed \& compact}}$

(Hint graph)

iv) X complete variety $\Longrightarrow s \in \Gamma(X, \mathcal{O}_X)$ constant. (Hint. $\Gamma(X, \mathcal{O}_X) = \operatorname{Mor}(X, \mathbb{A}^1)$) see Sec 2.3 notes

v) Deduce that affine varieties $\neq$ point, $\emptyset$ are never complete, and that the only global sections of a projective variety X are constant morphisms $X \longrightarrow \mathbb{A}^1$.

5) Note that any "commutative diagram" in a category $\mathcal{C}$ can be thought of as a functor $F: I \to \mathcal{C}$ where the objects of I are the positions i in the diagram (where you place some object $F(i) = C_i \in \mathcal{C}$), the morphisms of I are the arrows of the diagram (together with all identity morphs $i \to i$, and composites)

"inverse limit":

The limit $L = \varprojlim C_i \in \mathcal{C}$ (if exists) has morphs $L \xrightarrow{\pi_i} C_i$ s.t.

- compatible: $\forall (i \xrightarrow{\varphi} j) \in I$:
$$\begin{tikzcd} & C_i \arrow[d, "F(\varphi)"] \\ L \arrow[ur, "\pi_i"] \arrow[r, "\pi_j"'] & C_j \end{tikzcd}$$
- universal property:
$$\begin{tikzcd} & & C_i \arrow[d] \\ Z \arrow[r, dashed, "\exists!"] \arrow[urr, bend left, "\forall p_i"] \arrow[drr, bend right, "\forall p_j"'] & L \arrow[ur, "\pi_i"] \arrow[r, "\pi_j"'] & C_j \end{tikzcd}$$

"direct limit":

The colimit $D = \varinjlim C_i$ is defined by reversing arrows π_i, p_i (so $C_i \xrightarrow{\pi_i} D$).

EXAMPLE In sets, $\varprojlim C_i = \{(x_i) \in \prod C_i : x_i \xmapsto{F(i\to j)} x_j\}$, $\varinjlim C_i = \bigsqcup C_i / \langle x_i \sim x_j \text{ if } x_i \xmapsto{F(i\to j)} x_j \rangle$ $\longleftarrow$ generate an equivalence e.g. $x_i \sim x_j$, $x_j \sim x_k$ then declare $x_i \sim x_k$

i) What is the functor of points interpretation of $\varprojlim$, $\varinjlim$? (Hint. for $\varinjlim$ consider I^{op} and h^* not h_*)

ii) Explain briefly why the product, fiber product, gluing of sheaves are limits, and the coproduct, pushout, gluing of schemes are colimits (e.g. every scheme $= \varinjlim$ of its affine opens)

iii) Suppose f, g are adjoint functors ($\operatorname{Mor}_{\mathcal{D}}(fC, D) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(C, gD)$ bijection, functorial in C, D)

Show that left adjoints commute with colimits, right adjoints commute with limits:
- $g(\varprojlim C_i) = \varprojlim gC_i$
- $f(\varinjlim C_i) = \varinjlim fC_i$