C2.6 Introduction to Schemes

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Feedback and corrections are welcome!

EXERCISE SHEET 2

1) Recall in classical algebraic geometry, over a field k, projective n-space is $\mathbb{P}^n = (k^{n+1}\setminus\{0\})/k^*$ so identify $[a_0:a_1:\ldots:a_n] = [ta_0:ta_1:\ldots:ta_n]$ $\forall t\neq 0 \in k$

Notice $\mathbb{P}^n$ is covered by $n+1$ affine spaces $\mathbb{A}^n \cong \{[a_0:\ldots:a_{i-1}:1:a_{i+1}:\ldots:a_n]\}$

Projective n-space over R

i) Construct the scheme $\mathbb{P}^n_R$ by gluing $n+1$ copies of $\mathbb{A}^n_R = \mathrm{Spec}\, R[y_1,\ldots,y_n]$ where for the i-th copy you use $y_1 = \frac{x_0}{x_i}, y_2 = \frac{x_1}{x_i}, \ldots, y_n = \frac{x_n}{x_i}$ ← omit the case x_i/x_i (these generate a k-subalgebra of $S^{-1}R[x_0,\ldots,x_n]$ for $S=$ multiplicative set generated by $x_0,x_1,\ldots,x_n$)

ii) Show that a hom of rings $R\to S$ yields a natural map $\mathbb{P}^n_S \to \mathbb{P}^n_R$

iii) Construct $\mathbb{P}^n_U$ for any open subscheme $U\subseteq \mathrm{Spec}\, R$ ← (compare lecture notes on $\mathbb{A}^n_U$)

iv) Construct $\mathbb{P}^n_X \to X$: the projective n-space over any scheme X, and explain why a morphism $X\to Y$ induces a natural morphism $\mathbb{P}^n_X \to \mathbb{P}^n_Y$. Hint. Look at affine varieties.

2) $(X,\mathcal{O}_X)$ scheme, $s\in\mathcal{O}_X(U) \Rightarrow \{x\in U : s_x = 0\in\mathcal{O}_{X,x}\}$ open in U, need not be closed

$\Rightarrow \{x\in U : s(x) = 0 \in k(x)\}$ closed in U, need not be open

3) i) R_1,R_2 rings. Use natural projections $R_1\times R_2 \to R_i$ to show $\mathrm{Spec}\, R_1 \sqcup \mathrm{Spec}\, R_2 \cong \mathrm{Spec}\, R_1\times R_2$ ← categorical coproduct in Aff

Show that $\mathrm{Spec}(R_1\times R_2) = \{p_1\times R_2, R_1\times p_2 : p_1\in\mathrm{Spec}\, R_1, p_2\in\mathrm{Spec}\, R_2\}$.

ii) $(X,\mathcal{O}_X)$ scheme, U,V disjoint affine opens $\Longrightarrow U\cup V$ affine

iii) $(X,\mathcal{O}_X)$ irreducible $\Longleftrightarrow$ all affine opens are irreducible

iv) $\mathcal{O}_X(U)$ integral domain $\forall$ affine $U \Rightarrow X$ integral

v) X integral $\Longleftrightarrow$ irreducible & reduced (use Sheet 1)

Finally deduce: $\mathrm{Spec}\, R$ integral $\Leftrightarrow$ R is I.D.

Hint. First show X is irred. Then use ex. 2

Hint. ($\Leftarrow$) $X = C_1\cup C_2 = \cup U_i$ affine cover; is $U_i\cap U_j = \emptyset$ possible?

4) Consider the scheme $Y = \mathrm{Spec}\, k[x,y]/(f)$ where $f = y^2-x^2-x^3$. $k=$ field of characteristic $\neq 2$

i) Show that Y is an integral scheme.

ii) Draw a picture in $\mathbb{R}^2$ of the curve $f=0$. ← (just a sketch like in high-school)

Now consider the functor of points $h_Y(X)$ for the following test schemes X:

iii) $X = \mathrm{Spec}\, k[[x,y]]/(f)$ ← (power series in x,y)

Using the "natural choice" of $\alpha\in h_Y(X)$, show that $\alpha^{-1}(Y)$ is reducible.

use Fact (Newton's binomial thm) $(1+x)^r = 1 + rx + \frac{r\cdot(r-1)}{1\cdot 2}x^2 + \frac{r(r-1)(r-2)}{1\cdot 2\cdot 3}x^3 + \ldots \in k[[x,y]]$ provided the fractions exist $\in k$, and $r\in\mathbb{Q}$.

Hint $\mathbb{Z}\ni\binom{2n}{n} = \frac{(2n)!}{n!\,n!}$ and $\binom{2n}{n}\frac{1}{n+1} = \binom{2n}{n+1}\frac{1}{n}$.

iv) What would happen in (iii) for $X = \mathrm{Spec}\,\mathcal{O}_{Y,0}$? ← point $0\in Y$. Comment in view of picture (i).

5) An element e is called idempotent if $e^2=e$

Motivating example: in linear algebra the idempotent linear maps are precisely the projection maps onto vector subspaces

Note • In integral domains, only 0,1 are idempotents
• e idempotent $\Rightarrow$ $1-e$ idempotent

$X = \operatorname{Spec} R$

i) Show $X = D_e \sqcup D_{1-e}$ $\forall$ idempotent $e \in R$

Hint what is $e \in k(p)$? (called $e(p)$)

Example In 3(i), $\operatorname{Spec} R_1 \times R_2 = D_{(1,0)} \sqcup D_{(0,1)} = \operatorname{Spec} R_1 \sqcup \operatorname{Spec} R_2$

ii) $D_f \cap D_g = \emptyset \Leftrightarrow fg$ nilpotent (Example: $D_e \cap D_{1-e} = \emptyset$ in (i))

iii) $U \subseteq X$ is open and closed $\Leftrightarrow \exists !$ idempotent $e \in R$ with $U = D_e$

Hint $U = \mathbb{V}(I)$, $V = X \setminus U = \mathbb{V}(J)$, show $(IJ)^N = 0$ some N, "hence" $1 = 1^{2N} \in I^N + J^N$
$= (1-e)+e$

iv) Show (Connected component of $p \in X$) $= \mathbb{V}(\langle$idempotents $e \in R$ with $e(p) = 0 \in k(p)\rangle)$

use Fact (from topology): If a topspace X compact, has basis of compact opens, intersection of any two compact opens is compact $\Rightarrow$ connected component of $x \in X$ is $\bigcap\{$closed open $U \ni x\}$

Finally deduce: $\operatorname{Spec} R$ is connected $\Leftrightarrow$ 0,1 are only idempotents of R

6) A family of schemes is a morphism $X \xrightarrow{f} B$ of schemes.

Think of this as the collection of schemes $X_b = f^{-1}(b) = \operatorname{Spec} k(b) \times_B X$ ← fiber product: On affines this is the tensor product of algebras

A family of closed subschemes of Y over B is a closed subscheme $X \subseteq Y \times B$

$$\begin{array}{c} X \subseteq Y\times B \\ \searrow \quad \downarrow \text{project} \\ B \end{array}$$

i) Let $B = \operatorname{Spec} k[t]$ (k = field) $\quad B = \mathbb{A}^1_k$

$B^* = D_0 = \operatorname{Spec} k[t, t^{-1}]$ (basic open for $0 \in k[t]$) $\quad B^* = \mathbb{A}^1_k \setminus \{0\}$

$X^* = \mathbb{V}(x^2 - t^2) \subseteq \mathbb{A}^1_{B^*} = \operatorname{Spec} k[t, t^{-1}, x]$

Calculate the closure X of $X^* \subseteq \mathbb{A}^1_B = \operatorname{Spec} k[t,x]$ and the fiber X_0 ← Think of X_0 as the "limit" of X_b as $b \to 0$

ii) Repeat (i) for $X^* = \mathbb{V}(xy - t) \subseteq \mathbb{A}^2_{B^*} = \operatorname{Spec} k[t, t^{-1}, x, y]$

What pictures over $k = \mathbb{R}$ and $k = \mathbb{C}$ does this correspond to? ← only consider closed points for the picture

iii) For the family $X = \operatorname{Spec} \mathbb{Z}[x,y]/(x^2 - y^2 - 5) \longrightarrow B = \operatorname{Spec} \mathbb{Z}$ ← induced by obvious map on rings

What are the fibers $X_{(0)}, X_{(2)}, X_{(3)}, X_{(5)}$?

Show that this is a flat family. ← (the notes will help)

What happens if you replace $x^2 - y^2 - 5$ by $2x^2 - 2y^2 - 6$?