Exercise sheet 4*

(for the last class, at the beginning of Trinity Term)

Exercise 1.1. *Let $\phi : A \to B$ be a morphism of integral rings. Suppose that $\mathrm{Spec}(\phi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has dense image. Show that ϕ is injective.*

Exercise 1.2. *Let X be a noetherian scheme and let L, M be line bundles on X.*

(a) Suppose that L is ample. Show that for sufficiently large $n \geqslant 0$, the line bundle $L^{\otimes n} \otimes M$ is ample.

(b) Suppose that L and M are ample. Show that the line bundle $L \otimes M$ is ample.

Exercise 1.3. *Let L be a line bundle on $\mathbb{P}^1_{\mathbb{C}}$. Let $\sigma \in \Gamma(\mathbb{P}^1_{\mathbb{C}}, L)$ and suppose that $\sigma \neq 0$. Let $Z(\sigma) \hookrightarrow \mathbb{P}^1_{\mathbb{C}}$ be the zero scheme associated with σ. Let $V_\sigma := \Gamma(Z(\sigma), \mathcal{O}_{Z(\sigma)})$ where V_σ is viewed as a $\mathbb{C}$-vector space. Prove that $\dim_{\mathbb{C}}(V_\sigma)$ is independent of σ. It is called the* **degree** *of L.*

Exercise 1.4. *Let L, M be line bundles on $\mathbb{P}^1_{\mathbb{C}}$. Suppose that $\Gamma(\mathbb{P}^1_{\mathbb{C}}, L) \neq 0$ and that $\Gamma(\mathbb{P}^1_{\mathbb{C}}, M) \neq 0$. Suppose that L and M have the same degree. Prove that L is isomorphic to M as an $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$-module.*

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*For the course C2.6 Introduction to Schemes, Oxford, Hilary Term 2019