Noncommutative Rings, HT 2020
Problem Sheet 1

1. Let R be a ring containing an element x. Let the function $\mathrm{ad}_x : R \to R$ be defined by $\mathrm{ad}_x(r) = [x, r] = xr - rx$ for all $r \in R$. By using the binomial theorem, or otherwise, prove that

$$x^n r = \sum_{i=0}^{n} \binom{n}{i} \mathrm{ad}_x^i(r) x^{n-i} \quad \text{for all} \quad r \in R.$$

2. Let $\delta : R \to R$ be a *derivation*, meaning that δ is an additive function such that $\delta(rs) = \delta(r)s + r\delta(s)$ for all $r, s \in R$. Define the *skew-polynomial ring* $R[x; \delta]$ to be the quotient of the free ring generated by R and a formal variable X, by the ideal generated by elements of the form $Xr - rX - \delta(r)$:

$$R[x; \delta] := R\langle X\rangle / \langle Xr - rX - \delta(r) : r \in R\rangle.$$

There is a ring homomorphism $\varphi : R \to R[x; \delta]$ such that $\varphi(r)$ is the image of $r \in R\langle X\rangle$ in $R[x; \delta]$.

   (a) Suppose that $\psi : R \to S$ is a ring homomorphism, and that there is an element $\sigma \in S$ such that $\sigma\psi(r) = \psi(r)\sigma + \psi(\delta(r))$ for all $r \in R$. Let x denote the image of X in $R[x; \delta]$. Prove that there is a unique ring homomorphism $\theta : R[x; \delta] \to S$ such that $\psi = \theta \circ \varphi$ and $\theta(x) = \sigma$.

   (b) Let $V := R^{\mathbb{N}}$ be the abelian group of infinite sequences $(v_i) = (v_0, v_1, v_2, \cdots)$ of elements of R, and let $S := \mathrm{End}_{\mathbb{Z}}(V)$ be its endomorphism ring. Define $\psi : R \to S$ by the rule $\psi(r)((v_i)) = (rv_i)$, and let $\sigma \in S$ be given by $\sigma((v_i)) = (v_{i-1} + \delta(v_i))$ where $v_{-1} := 0$. Prove that ψ is an injective ring homomorphism, and that $\sigma\psi(r) = \psi(r)\sigma + \psi(\delta(r))$ for all $r \in R$.

   (c) Prove that $\{1, x, x^2, \cdots\}$ is a basis for the left R-module $R[x; \delta]$.

3. Write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n} \in A_n(k)$ for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.

   (a) Deduce from Theorem 1.12 that $A_n(k)$ is left Noetherian.

   (b) Prove that $\{x^\alpha y^\beta : \alpha, \beta \in \mathbb{N}^n\}$ is a basis for $A_n(k)$ as a k-vector space.

   (c) Prove that $A_n(k) \to \mathrm{End}_k(k[x_1, \ldots, x_n])$ is injective if and only if $\mathrm{char}(k) = 0$.

4. (a) Let R be a ring, and let N be a submodule of an R-module M. Prove that M is Noetherian if and only if N and M/N are Noetherian.

   (b) Suppose that M is the sum of its Noetherian submodules $M_1, \ldots, M_n$. Prove that M is also Noetherian.

   (c) Let R be a left Noetherian subring of a ring S which is finitely generated as a left R-module. Prove that S is also left Noetherian.

   (d) Let G be a group which contains a polycyclic subgroup of finite index, and suppose that R is left Noetherian. Prove that RG is also left Noetherian.