Noncommutative Rings
Problem Sheet 2

1. A ring R is called a *domain* if $xy = 0$ implies $x = 0$ or $y = 0$ for any $x, y \in R$. A filtration $(R_i)_{i\in\mathbb{Z}}$ is called *separated* if $\bigcap_{i\in\mathbb{Z}} R_i = \{0\}$.

   (a) Suppose that $(R_i)_{i\in\mathbb{Z}}$ is a separated filtration on a ring R such that $\operatorname{gr} R$ is a domain. Prove that R is also a domain.

   (b) Find an example of a separated filtration on a domain R such that $\operatorname{gr} R$ is not a domain.

2. (a) Show that the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right Noetherian, but not left Noetherian.

   (b) Find an example of a ring which is left Noetherian but not right Noetherian.

   (c) Let $R = k\langle x, y\rangle$ be the free algebra on two generators. Prove that R is not left Noetherian.

3. The *idealiser subring* of a right ideal I of R is defined by $N(I) := \{r \in R : rI \subseteq I\}$.

   (a) Show that $N(I)$ is the largest subring of R containing I as a two-sided ideal.

   (b) Prove that $N(I)/I$ is isomorphic to $\operatorname{End}((R/I)_R)$.

4. An element $e \in R$ is an *idempotent* if $e^2 = e$. Suppose $e \in R$ is a central idempotent.

   (a) Show that $1 - e$ is also a central idempotent.

   (b) Show that eR is a ring with the same multiplication as in R, but with identity element e.

   (c) Show that R is isomorphic to the direct product of rings $eR \times (1 - e)R$.

   (d) What happens if e is not assumed to be central?

5. Let $M_n(R)$ be the $n \times n$ matrix ring with coefficients in R.

   (a) Show that any two-sided ideal of $M_n(R)$ has the form $M_n(I)$ for some two-sided ideal I of R.

   (b) We say that the ring R is *simple* if it has no nontrivial proper two-sided ideals. Prove that $M_n(R)$ is a simple ring if and only if R is simple.

   (c) Prove that $M_n(D)$ is a simple ring for any division ring D.

6. Let M be a left R-module.

   (a) Suppose that $R = M_n(D)$ for some division ring D and $n \geq 1$. Show that M is Artinian if and only if it is Noetherian.

(b) Suppose now that $R \cong R_1 \times \cdots \times R_n$ for some rings $R_1, \ldots, R_n$. Show that M is isomorphic to $M_1 \times \cdots \times M_n$ for some left R_i-modules M_i.

(c) Suppose further that R is the direct product of finitely many matrix rings over division rings. Show that M is Artinian if and only if it is Noetherian.

7. Let R be a left Artinian ring and let $x \in R$ be such that $sx = 0$ implies $s = 0$. Prove that x is a unit in R. Deduce that a left Artinian domain must be a division ring.