Noncommutative Rings
Problem Sheet 3

Throughout this sheet, A will denote a ring.

1. Let S be a left localisable subset of A, and let M be a left A-module. Prove that
   (a) $t_S(M)$ is an A-submodule of M,
   (b) $t_S(M/t_S(M)) = 0$,
   (c) the map $M \to S^{-1}M$ given by $m \mapsto 1\backslash m$ has kernel $t_S(M)$,
   (d) $S^{-1}M = 0$ if and only if $M = t_S(M)$.

2. Let S be a multiplicatively closed subset of A, let I be a two-sided ideal in A and let $\overline{S}$ be the image of S in $\overline{A} := A/I$. Prove that $\overline{S}$ is a left Ore set in $\overline{A}$ if S is a left Ore set in A. Is the converse true?

3. Let P be a two-sided ideal in A such that A/P is a domain. Suppose that $S := A \backslash P$ is left localisable. Prove that $S^{-1}P$ is the unique maximal left ideal of $S^{-1}A$.

4. Let R be a commutative domain, and suppose that $A = M_n(R)$ for some $n \geq 1$.
   (a) Show $s \in A$ is regular if and only if $\det(s) \neq 0$.
   (b) Prove that regular elements form a left Ore set in A.

5. Let k be a field, let V be a countably-infinite dimensional k-vector space and let $A = \mathrm{End}_k(V)$. Show that $S := \{s \in A : s \text{ is surjective}\}$ is multiplicatively closed, and that $S^{-1}A$ is the zero ring.

6. Show that a maximal two-sided ideal in A is left primitive, and a left primitive ideal in A is prime. Find an example of a ring A, and a prime ideal P in A, such that P is not left primitive.

7. (a) Let $A = k[x, y, z]$ be the polynomial ring in three variables over a field k, and let $I = (xy, yz, zx)$. Find $\min(I)$, and justify your answer.
   (b) Suppose that A is a commutative Noetherian graded ring, and let I be a graded ideal in A. Prove that $\sqrt{I}$ is also a graded ideal.

8. Suppose that A is commutative and Noetherian.
   (a) If M is a finitely generated A-module and $I = \mathrm{Ann}_A(M)$, show that A/I is isomorphic to an A-submodule of M^n for some $n \in \mathbb{N}$.
   (b) If $J \lhd A$ and d is a dimension function for A, prove that $d(A/J) = d(A/J^m)$ for all $m \geq 1$.
   (c) Prove that a dimension function for A is completely determined by the values it takes on modules of the form A/P where $P \in \mathrm{Spec}(A)$.