Noncommutative Rings

Problem Sheet 4

1. Let $A = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and let $P = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$. Show that P is a prime ideal in A. Also, show that $S := A \backslash P$ is multiplicatively closed but is not a right Ore set. Prove that S *is* a left localisable subset of A and that $S^{-1}A \cong \mathbb{Q}$.

2. Suppose that A is left Noetherian, and let S be a left localisable subset of A.

   (a) Prove that $Q := S^{-1}A$ is also left Noetherian.

   (b) Show that if I is a two-sided ideal in A then $Q \cdot I$ is also a two-sided ideal in Q.

   (c) Suppose further that A is also right Noetherian, and that P is a prime ideal in A such that $P \cap S = \emptyset$. Show that $Q \cdot P$ is a prime ideal in Q.

3. Let A be a filtered ring and let M be a filtered left A-module.

   (a) Show that $\widetilde{M}/t\widetilde{M}$ is isomorphic to $\operatorname{gr} M$ as a left $\operatorname{gr} A$-module.

   (b) Viewing M as a left $\widetilde{A}$-module via the isomorphism $\widetilde{A}/(t-1)\widetilde{A} \cong A$ from Lemma 4.20(2), show that $\widetilde{M}/(t-1)\widetilde{M}$ is isomorphic to M as a left $\widetilde{A}$-module.

4. (a) Verify that the commutator bracket on a ring A is a Poisson bracket.

   (b) Let k be a field. Suppose that $\{,\}$ is a Poisson bracket on the polynomial ring $A = k[x_1, \ldots, x_n]$ such that $\{k, A\} = 0$. Prove that $\{,\}$ is completely determined by its values on the x_i's.

   (c) Let A be a filtered ring such that $\operatorname{gr} A$ is commutative, and let $\{,\}$ be the induced Poisson bracket on $\operatorname{gr} A$. Show that $\operatorname{gr} I$ is closed under $\{,\}$ for any left ideal I in A.

   (d) Find an example of a filtered ring A and a graded ideal J in $\operatorname{gr} A$ such that $\operatorname{gr} A$ is commutative and $\{J, J\} \subseteq J$ but $\{\sqrt{J}, \sqrt{J}\} \nsubseteq \sqrt{J}$.

5. Let B be a left Noetherian ring, and let $t \in B$ be a central regular element. By considering the ring $(t^{\mathbb{N}})^{-1}B$ or otherwise, show that for any left ideal I of B there is an integer n such that $I \cap t^n B \subseteq tI$.

6. Let $n \geq 1$, and let k be a field of characteristic zero. Show that there are no $n \times n$ matrices X, Y with entries in k that satisfy the relation $YX - XY = 1$. What happens if the characteristic of k is positive?

7. Let R be a filtered ring, let M be a filtered left $R-$module with filtration $(M_i)_{i \in \mathbb{Z}}$ and let N be a submodule of M. Equip N with the *subspace filtration* $N_i := N \cap M_i$, and equip M/N with the *quotient filtration* $(M/N)_i := (M_i + N)/N$. Show that

(a) there is an injective $\operatorname{gr} R$-module homomorphism $\alpha : \operatorname{gr} N \to \operatorname{gr} M$,

(b) there is a surjective $\operatorname{gr} R$-module homomorphism $\beta : \operatorname{gr} M \to \operatorname{gr}(M/N)$,

(c) $\ker \beta = \operatorname{Im} \alpha$.

8. Let $A = A_n(k)$ be the Weyl algebra, and let r be an integer such that $n \leqslant r \leqslant 2n$. Give an example of a cyclic A-module M such that $d(M) = r$. Justify your answer.