C2.3 Representations of semisimple Lie algebras

Mathematical Institute, University of Oxford
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Problem Sheet 1

1. Let V, W be $\mathfrak{g}$-modules and suppose that V is finite dimensional. Exhibit an isomorphism

$$V^* \otimes_{\mathsf{k}} W \cong \mathrm{Hom}_{\mathsf{k}}(V, W)$$

of $\mathfrak{g}$-modules.

2. Let V be a finite dimensional complex vector space and $\mathfrak{g} = \mathfrak{sl}(V)$.

(i) Verify that $V \otimes V = S^2(V) \bigoplus \bigwedge^2 V$ is a decomposition as $\mathfrak{g}$-representations.

(ii) Show that $S^2(V)$ and $\bigwedge^2 V$ are simple $\mathfrak{g}$-modules.

(iii) Let $\mathfrak{g}' \subset \mathfrak{g}$ be an orthogonal (or symplectic) Lie algebra with respect to the nondegenerate bilinear symmetric (or skew-symmetric) form B. Then from (ii), $S^2(V)$ and $\bigwedge^2 V$ are $\mathfrak{g}'$-modules. Are they simple $\mathfrak{g}'$-modules?

3. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k of characteristic 0.

(i) Show that every derivation D of $\mathfrak{g}$ extends to a unique derivation D' of $U(\mathfrak{g})$.

(ii) If $D = \mathrm{ad}_{\mathfrak{g}}(x)$, show that $D'(u) = xu - ux$ for all $u \in U(\mathfrak{g})$.

(iii) Notice that there exists a unique derivation D'' of $S(\mathfrak{g})$ which extends D. Show that $D'' \circ \phi = \phi \circ D'$, where $\phi : S(\mathfrak{g}) \to U(\mathfrak{g})$ is the symmetrizing isomorphism

$$\phi(x_1 x_2 \cdots x_m) = \frac{1}{m!} \sum_{\sigma \in S_m} x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(m)}.$$

(iv) Recall that $\mathfrak{g}$ acts on both $S(\mathfrak{g})$ and $U(\mathfrak{g})$ via 'extending' the adjoint action of $\mathfrak{g}$ on itself to tensor products. Prove that ϕ is an isomorphism of $\mathfrak{g}$-modules.

(v) Deduce that there is a linear isomorphism between the spaces of $\mathfrak{g}$-invariants $S(\mathfrak{g})^{\mathfrak{g}} \cong U(\mathfrak{g})^{\mathfrak{g}}$. Check that the latter space is in fact the centre of $U(\mathfrak{g})$. [1]

4. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and recall the finite dimensional irreducible $\mathfrak{g}$-modules $V(n)$ of dimension $n + 1$ constructed in the lectures. (Here, $n \geq 0$ is an integer.)

(i) Verify that $V(n) \cong S^n(V(1))$ as $\mathfrak{g}$-modules, where S^n denotes the n-th symmetric power.

(ii) Decompose the tensor product $V(2) \otimes V(3)$ into a direct sum of simple $\mathfrak{g}$-modules.

(iii) For $n \geq m \geq 0$, prove that

$$V(n) \otimes V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m).$$

(iv) Find the decomposition of $S^n(V(2))$ into a direct sum of irreducible representations.

[*Hint: use the eigenspace decomposition with respect to the action of $h \in \mathfrak{g}$.*]

5. Suppose the characteristic of the field k is $p > 0$, and let $\mathfrak{g} = \mathfrak{sl}(2, \mathsf{k})$ and let $V(n)$ be the $\mathfrak{g}$-modules of dimension $n+1$ as before. Prove that $V(n)$ is irreducible as long as $n < p$, but reducible when $n = p$.

6 (*optional*)**.** Suppose that we know that every $\mathfrak{g}$-module of length 2 is completely reducible. Prove that every $\mathfrak{g}$-module of finite length is completely reducible. (This is a general lemma that holds in every abelian category.)

[*Hint: use induction on the length of the module. If V is a reducible module of finite length, let S be a simple submodule of V, and consider $Q = V/S$.*]

[1] Warning: this linear isomorphism is not an isomorphism of algebras, even though it is a linear isomorphism and both algebras are commutative!