C2.3 Representations of semisimple Lie algebras

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Hilary Term 2020

Problem Sheet 3

All Lie algebras are defined over an algebraically closed field k of characteristic zero. Unless otherwise stated, $\mathfrak{g}$ will denote a semisimple Lie algebra with maximal toral subalgebra $\mathfrak{h}$ and root system Φ.

1. Let $\Phi^+ = \{\alpha_1, \ldots, \alpha_m\}$ be the set of positive roots of $\mathfrak{g}$. Define the *Kostant partition function* $K : \mathfrak{h}^* \to \mathbb{N}$ as follows:

$$K(\lambda) := |\{(i_1, \ldots, i_m) \in \mathbb{N}^m : i_1\alpha_1 + \cdots + i_m\alpha_m = \lambda\}| \quad \text{for all} \quad \lambda \in \mathfrak{h}^*.$$

Thus $K(\lambda)$ is the number of ways in which λ can be written as a linear combination of positive roots with nonnegative integer coefficients.

Let $M(\lambda)$ be a Verma module. Prove that $\dim M(\lambda)_\mu = K(\lambda - \mu)$, for all $\mu \in \mathfrak{h}^*$.

2. Let $\mathfrak{g} = \mathfrak{sl}(n)$ acting naturally on $V := \mathsf{k}^n$. For $1 \leq m \leq n-1$, define $U_m := \bigwedge^m V$.

(a) Show that each U_m is a simple $\mathfrak{g}$-module and determine its highest weight.

(b) Deduce that any simple finite dimensional $\mathfrak{g}$-module appears as a summand of

$$S^{m_1}(U_1) \otimes S^{m_2}(U_2) \otimes \cdots \otimes S^{m_{n-1}}(U_{n-1})$$

for some nonnegative integers m_i.

3. Let $\mathfrak{g} = \mathfrak{sl}(2)$ and let $M(\lambda)$ be the Verma module for some $\lambda \in \mathsf{k}$.

(a) Show from first principles, that if λ is a nonnegative integer then there exists an injective map $\phi : M(-\lambda - 2) \hookrightarrow M(\lambda)$ whose cokernel $L(\lambda) := M(\lambda)/\mathrm{Im}\, \phi$ is irreducible.

(b) Show that $M(\lambda) \otimes M(\mu)$ is not in category $\mathcal{O}$ for any $\mu \in \mathsf{k}$.

(c) Determine a composition series for $M(\lambda) \otimes L(\mu)$ when $\mu \in \mathbb{N}$.

4. Consider the shifted W-action on $\mathfrak{h}^*$: $w \bullet \lambda = w(\lambda + \rho) - \rho$, where $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$.

(a) What are the shifted W-orbits of $\mathfrak{h}^*$ with the minimal, respectively maximal number of elements?

(b) Suppose $\mathfrak{g} = \mathfrak{sl}(3)$ and $\Delta = \{\alpha, \beta\}$. Calculate the shifted W-orbit of $\lambda \in P$ when:
(i) $\lambda = 0$, (ii) $\lambda = \alpha$, and (iii) $\lambda = \omega_1$ is a fundamental weight.

5. A *Chevalley anti-involution* of $\mathfrak{g}$ is a k-linear map $\tau : \mathfrak{g} \to \mathfrak{g}$ satisfying the following conditions:

- τ is an *anti-involution* of $\mathfrak{g}$: $\tau^2 = \mathrm{id}$ and $\tau([x, y]) = [\tau(y), \tau(x)]$, for all $x, y \in \mathfrak{g}$,
- τ restricts to the identity map on $\mathfrak{h}$,
- $\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for all $\alpha \in \Phi$.

(a) Show that the transpose map $A \mapsto A^T$ on matrices is a Chevalley anti-involution when $\mathfrak{g} = \mathfrak{sl}(n)$.

Suppose that τ is a Chevalley anti-involution of $\mathfrak{g}$.

(b) Show that τ extends to an anti-automorphism τ of $U(\mathfrak{g})$.

(c) Prove that $\tau : U(\mathfrak{g}) \to U(\mathfrak{g})$ fixes $Z(\mathfrak{g})$ pointwise.