Sheet 4

Question 1

The proof is essentially the same as for L.

The absoluteness result needed is of course: if B, C are transitive classes and $A \in B$ then the class function $L[A] : \mathrm{On} \to V$ is absolute for B, C.

For choice, instead of starting with the trivial well-order on L_0, you note that if $A \subseteq \mathrm{On}$ then $TC(\{A\}) = \{A\} \cup \sup A$ so that $L[A]_0$ can be well-ordered by taking the natural well-order on $\sup A \in \mathrm{On}$ and following this by $\{A\}$ (or having $\{A\}$ first followed by $\sup A$)

To see that if $A \subseteq \omega$ then $L[A] \models \mathbf{CH}$, you follow the proof that $L \models \mathbf{CH}$.

Question 2

If $L_\alpha = V_\alpha$ then $|L_\alpha| = |V_\alpha|$. But since $V = L$ we have **GCH** so that $|V_\alpha| = \aleph_\alpha$, giving $\aleph_\alpha = |\alpha| \leq \alpha$. But by a very easy induction on On we have $\alpha \leq \aleph_\alpha$ for all $\alpha \in \mathrm{On}$. Hence $\alpha = \aleph_\alpha$ as required.

For the converse, assume that $\alpha = \aleph_\alpha$. $L_\alpha \subseteq V_\alpha$ is always true (see lecture notes). So assume that $x \in V_\alpha$. Since α is a cardinal, it is a limit ordinal, so $x \in V_\beta$ for some $\beta \in \alpha$. Hence $|TC(\{x\})| \leq |V_\beta| = \aleph_\beta < \aleph_\alpha$ (using **GCH** for the $=$). Thus $x \in H_{\aleph_\alpha}$ and from the proof that $V = L \to \mathbf{GCH}$ we have $H_{\aleph_\alpha} = L_{\aleph_\alpha}$ so $x \in L_{\aleph_\alpha} = L_\alpha$.

To construct ordinals α such that $\aleph_\alpha = \alpha$, we employ recursion: define $F : \mathrm{On} \to \mathrm{On}; \beta \mapsto \aleph_\beta$. This is weakly increasing so by a previous sheet has arbitrarily large fixed points. Note that this only gives singular solutions (in fact, solutions of countable cofinality).

Question 3

That $cf(\alpha)$ is regular follows from $\mathrm{cf}\mathrm{cf}\alpha = \mathrm{cf}\alpha$.

Now assume that $\kappa \in \mathrm{Card}$. If κ^+ is not regular, then there is some ordinal $\beta < \kappa^+$ and an unbounded $f : \beta \to \kappa^+$. But then $|\beta| \leq \kappa$ since κ^+ is a cardinal, so there is an unbounded $g : \kappa \to \kappa^+$. For each $\alpha \in \kappa$, $|g(\alpha)| \leq \kappa$ so that $\sup g = \bigcup_{\alpha \in \kappa} g(\alpha)$ has cardinality $\leq \kappa \otimes \kappa = \kappa < \kappa^+$. Hence g cannot be unbounded.

Question 4

For the first part, it is enough to show that $\kappa^{\mathrm{cf}\kappa} > \kappa$. By a result from the lecture notes, we have a weakly increasing unbounded $f : \mathrm{cf}\kappa \to \kappa$. We then apply König's inequality to the $f(\alpha) < \kappa$ to obtain

$$\kappa = \sup f \leq \sum_{\alpha \in \mathrm{cf}\kappa} f(\alpha) < \sum_{\alpha \in \mathrm{cf}\kappa} \kappa = \kappa^{\mathrm{cf}\kappa}.$$

Now assume that $\lambda < cf(\kappa)$. We define an injection from $\{f : \lambda \to \kappa\}$ into $\bigcup_{\alpha \in \kappa} \{f : \lambda \to \alpha\}$ as follows: for each $f : \lambda \to \kappa$ we must have $f[\lambda]$ bounded in

κ (since κ is a cardinal) and thus we have a minimal $\alpha_f < \kappa$ such that $f[\lambda] \subseteq \alpha_f$. We then send f to $f : \lambda \to \alpha_f$.

Thus

$$\kappa^\lambda \leq \sum_{\alpha \in \kappa} \left|\alpha^\lambda\right| \leq \kappa \otimes \sup \left|\alpha^\lambda\right|.$$

We next show that $\alpha \in \kappa$ implies $\left|\alpha^\lambda\right| \leq \kappa$ and hence the result follows. Since $\left|\alpha^\lambda\right| = |\alpha|^\lambda$ we may assume that $\alpha < \kappa$ and α is a cardinal. But for these $\alpha < 2^\alpha$ so that

$$\alpha^\lambda \leq [2^\alpha]^\lambda = 2^{\alpha \otimes \lambda} = 2^{\max\{\alpha,\lambda\}} \leq \kappa$$

by the assumption and the fact that $\max \alpha, \lambda < \kappa$.

Now assume **GCH**. As above (and without **GCH**) if $\lambda \leq \kappa$ then $\kappa^\lambda \leq [2^\kappa]^\lambda = 2^\kappa$. Applying **GCH** then gives $\kappa^\lambda \leq \kappa^+$.

Of course, for any $\lambda \geq 1$ we have $\kappa \leq \kappa^\lambda$ giving the result.

Question 5

This is similar to a question from the previous sheet. We define recursively for $n \in \omega$, $\alpha_0 = \alpha$ and $\alpha_{n+1} = \sup g[\alpha_n]$. Since κ is a cardinal, if $\alpha_n \in \kappa$ then $\sup g[\alpha_n] \in \kappa$, so all $\alpha_n \in \kappa$ (by induction). Since κ is regular uncountable this implies $\beta = \sup \alpha_n \in \kappa$. This β works since if $\delta \in \beta$ then $\delta \in \alpha_n$ for some $n \in \omega$ and hence $g(\delta) \in g[\alpha_n] = \alpha_{n+1} \subseteq \beta$.

Question 6

(i): By induction on $\alpha < \kappa$ we show $|V_\alpha| < \kappa$: This is clear for finite ordinals and for ω. If $|V_\alpha| < \kappa$ then $|V_{\alpha+1}| = 2^{|V_\alpha|} < \kappa$ by assumption. If $\gamma < \kappa$ is a limit ordinal then $V_\gamma = \bigcup_{\beta<\gamma} V_\beta$ is a union of $< \kappa$ many sets of size $< \kappa$, so by regularity of κ has size $< \kappa$.

(ii): Since $\kappa \subseteq V_\kappa$ (some previous sheet) we must have $\kappa \leq |V_\kappa|$. But now V_κ is the union of κ many sets of size $\leq \kappa$ (by (i)) so has size at most $\kappa.\kappa = \kappa$.

(iii): Suppose $\phi(x, y, \vec{v})$ is a formula, $\vec{a} \in V_\kappa^n$ and

$$V_\kappa \models \forall x \forall y, y' \ (\phi(x, y, \vec{a}) \wedge \phi(x, y', \vec{a}) \to y = y'$$

Write y_x for the unique $y \in V_\kappa$ such that $\phi(x, y_x, \vec{a})$ (if it exists) and $y_x = \emptyset$ if no such $y \in V_\kappa$ exists (depending on your precise formulation of **Replacement** you might not need this last bit).

Fix $d \in V_\kappa$ and apply **Replacement** with $\psi(x, y, \vec{v}) \equiv y \in V_\kappa \wedge \phi(x, y, \vec{v})^{V_\kappa}$ to obtain $z = \{y_x : x \in d\} \in V$. But $d \in V_\kappa$, κ is a limit ordinal, so $d \in V_\alpha$ thus $d \subseteq V_\alpha$ for some $\alpha < \kappa$ and hence $|d| < \kappa$. Also for each y_x we have $y_x \in V_\kappa$ so we can find $\alpha_x < \kappa$ with $y_x \in V_{\alpha_x}$. Then $\alpha = \sup \{\alpha_x : x \in d\} = \bigcup_{x \in d} \alpha_x$ is a $< \kappa$ union of sets of size $< \kappa$, so $\alpha < \kappa$ by regularity of κ and hence $\alpha + 1 < \kappa$ as κ is a limit ordinal. Hence $z \subseteq V_\alpha \in V_{\alpha+1} \subseteq V_\kappa$. It is now standard to verify that $V_\kappa \models z = \{y_x : x \in d\}$ as required.