Definition 1. A relation R on a class A is well-founded if and only if

$$\forall x \subseteq A \ \ [x \neq \emptyset \to \exists m \in x \ \forall b \in x \ \neg [bRm]]$$

(i.e. every non-empty subset of A has an R-minimal element).

Definition 2. A relation R on a class A is set-like if and only if

$$\forall x \in A \ \exists z \ \forall t \ \ [t \in z \leftrightarrow tRx]$$

Lemma 1. *Suppose R is a set-like relation on a class A. Then $\mathrm{pred}_R \colon A \to U$ given by $\mathrm{pred}_R(x) = \{t \in A : tRx\}$ is a class function on A.*

Proof. By **Extensionality** and the definition of being set-like. □

Lemma 2. *Suppose R is a well-founded, set-like relation on a class A. Then $R^\star$ given by*

$$xR^\star y \equiv \exists n \in \omega \ \exists f \colon n+2 \to A \ \ [f(0) = x \wedge f(n+1) = y \wedge \forall m \in n+1 \ \ [f(m)Rf(m+1)]]$$

is a well-founded, set-like relation on A that is transitive and contains R.

Proof. Clearly it is a relation on A.

If xRy then $n = 0$ and $f = \{\langle 0, x\rangle, \langle 1, y\rangle\}$ witnesses $xR^\star y$ so $R \subseteq R^\star$.

For $x, y \in A$ such that $xR^\star y$, we write $p(x, y)$ for the smallest $n \in \omega$ which witnesses $xR^\star y$.

We will show transitivity (i.e. $\forall x, y, z \in A \ \ [xR^\star y \wedge yR^\star z \to xR^\star z]$) by induction on $p(y, z)$. So assume this is true provided $p(y, z) < n$ and that $p(y, z) = n$. Let g be a witness of $yR^\star z$ with $dom(g) = n + 2$. The special case $n = 0$ is easy to handle manually. Otherwise, note that $p(y, g(n)) < n$ so that by inductive hypothesis there is $m \in \omega$ and $f \colon m + 2 \to A$ witnessing $xR^\star g(n)$. Then $m + 1$ together with $f \cup \{\langle m + 2, z\rangle\}$ witnesses $xR^\star z$.

Next we show that $R^\star$ is set-like. Fix $x \in A$. By induction on $n \in \omega$, we can see that

$$P_n = \{z \in A : zR^\star x \wedge p(z, x) \leq n + 1\} = \bigcup \{\mathrm{pred}_R(y) : y\mathbb{R}^\star x \wedge p(y, x) \leq n\}$$

is a set (by **Union** and **Replacement**) and (another application of **Union** and **Replacement**) so is $\mathrm{pred}_{R^\star}(x) = \bigcup \{P_n : n \in \omega\}$.

Finally for well-foundedness, let $C \subseteq A$ and $C \neq \emptyset$ and assume that x has no $R^\star$-minimal element. Let $x_0 \in C$ and

$$a = \{x \in \mathrm{pred}_{R^\star}(x_0) \cup \{x_0\} : \exists y \in C \ yR^\star x\}$$

(noting that this is a set by **Separation** and the fact that $R^\star$ is set-like.)

Since $x_0 \in C$ is not $R^\star$-minimal, $x_0 \in a$, so $a \neq \emptyset$. Since R is well-founded, a has an R-minimal element m. By definition of a, we can find $y \in C$ such that $yR^\star m$. Since $mR^\star x_0$ (or $m = x_0$) and $R^\star$ is transitive, we obtain $yR^\star x_0$ and since C has no $R^\star$-minimal element there must be $y' \in C$ with $y'R^\star y$, giving $y \in a$. Find an $n \in \omega$ and $f \colon n + 2 \to A$ witnessing $yR^\star m$. Now $f(n)R^\star m$ and $mR^\star x_0$ or $m = x_0$ given $f(n)R^\star x_0$ by transitivity of $R^\star$. Also $yR^\star f(n)$ or $y = f(n)$ so that $f(n) \in a$. But we also have $f(n)Rm$, contradicting R-minimality of m in a. □

Definition 3. Suppose R is a well-founded, set-like relation on U and $F\colon U \times U \to U$ is a class function.

We define

$$\begin{aligned}\psi_{F,R}(a,g) \equiv & g \text{ is a function on } \{a\} \cup \text{pred}^\star(a) \wedge \\ & \forall x \in \mathit{dom}(g)\ g(x) = F\left(x, g|_{\text{pred}(x)}\right)\end{aligned}$$

and

$$G_{F,R} \equiv \{\langle a,b\rangle : \exists g\ \psi_{F,R}(a,g) \wedge b = g(a)\}$$

Theorem 1. *Suppose R is a well-founded, set-like relation on U.*

If $F\colon U \times U \to U$ is a class function then, writing G for $G_{F,R}$, G is a class function on U such that

$$\forall x\ G(x) = F(x, G|_{\text{pred}(x)})$$

and for every class function H on U such that $\forall x\ H(x) = F(x, H|_{\text{pred}(x)})$ we have $\forall x\ G(x) = H(x)$.

We first prove a lemma (this is sort of the 'internal' version of the theorem 'up to any a').

Lemma 3. *Suppose R is a well-founded, set-like relation on Y.*

If $F\colon U \times U \to U$ is a class function then

$$\forall a\ \exists! g\ \psi_{F,R}(a,g).$$

Proof. Suppose this is not the case. Fix $x_0 \in U$ such that $\neg\exists! g\ \psi_{F,R}(x_0,g)$ and let a be $R^\star$-minimal in $\{y \in \{x_0\} \cup \text{pred}^\star(x_0) : \neg\exists! g\ \psi_{F,R}(y,g)\}$ (noting that this is a set and contains x_0 so is non-empty). Note that a is then $R^\star$-minimal such that $\neg\exists! g\ \psi_{F,R}(a,g)$.

Case $\neg\exists g\ \psi_{F,R}(a,g)$: Note that by $R^\star$-minimality of a, we have

$$\forall y \in \text{pred}^\star(a)\ \ \exists! g\ \psi_{F,R}(y,g)$$

so that

$$f = \{g : \exists y \in \text{pred}^\star(a)\ \ \psi_{F,R}(y,g)\}, \quad h = \bigcup f$$

are sets (by **Replacement** and **Union**). Note that if $g, g' \in f$ and $y \in \mathit{dom}(g) \cap \mathit{dom}(g')$ then $\hat{y} = \{y\} \cup \text{pred}^\star(y) \subseteq \mathit{dom}(g) \cap \mathit{dom}(g')$ and that $\psi_{F,R}(y, g|_{\hat{y}})$ as well as $\psi_{F,R}(y, g'|_{\hat{y}})$. Further $yR^\star a$ so that by $R^\star$-minimality of a we must have $g|_{\hat{y}} = g'|_{\hat{y}}$ and hence $g(y) = g'(y)$. Thus h is a function.

Next for $y \in \text{pred}^\star(a)$ find g such that $\psi_{F,R}(y,g)$ so that $g \subseteq h$ and hence $y \in \mathit{dom}(g) \subseteq \mathit{dom}(h)$.

Thus h is a function on $\text{pred}^\star(a)$ and we note that

$$g = h \cup \left\{\left\langle a, F(a, h|_{\text{pred}(a)})\right\rangle\right\}$$

is a function on $\{a\} \cup \text{pred}^\star(a)$ such that $\psi_{F,R}(a,g)$, a contradiction.

Case $\exists g, g' \ \ [\psi_{F,R}(a, g) \wedge \psi_{F,R}(a, g') \wedge g \neq g']$**:** By $R^\star$-minimality of a, $\forall y \in \operatorname{pred}^\star(a) \ \ g(y) = g'(y)$ and in particular $g|_{\operatorname{pred}(a)} = g'|_{\operatorname{pred}(a)}$. But then $g(a) = F(a, g|_{\operatorname{pred}(a)}) = F(a, g'|_{\operatorname{pred}(a)}) = g'(a)$ so that $g = g'$, a contradiction. □

Proof of the General Recursion Theorem. First, if $x \in U$ suppose that there are $y, y' \in U$ such that $\langle x, y\rangle, \langle x, y'\rangle \in G_{F,R}$. Find g, g' such that $\psi_{F,R}(x, g)$ and $g(x) = y$ and $\psi_{F,R}(x, g')$ and $g'(x) = y'$. Then by the Lemma $g = g'$ so that $y = g(x) = g'(x)y'$. Thus $G_{F,R}$ is a class function on some class A.

Next if $x \in U$ then there is (by the Lemma) g such that $\psi_{F,R}(x, g)$ so that $\langle x, g(x)\rangle \in G_{F,R}$. Hence $G_{F,R}$ is a class function on U.

That $G(a) = F(a, G|_{\operatorname{pred}(a)})$ follows from the construction of G and the method in the previous Lemma.

Similarly the 'uniqueness' is an easy consequence of the previous Lemma by considering a '$R^\star$-minimal' counterexample. □