Gödel Incompleteness Theorems: Solutions to sheet 0

1. (i) Show that the following is logically valid in any non-empty domain, where x is not free in G.

$$((\forall x\, F(x) \to G) \leftrightarrow \exists x\, (F(x) \to G)).$$

The implication $(\forall x\, F(x) \to G)$ *is false in a structure iff* $\forall x\, F(x)$ *is true and* G *is false, if and only if for all* a *in structure,* $F(a)$ *is true, and* G *is false (recall that* x *does not occur free in* G*), if and only if for all* a*,* $(F(a) \wedge \neg G)$ *is true, if and only if* $\forall x\, (F(x) \wedge \neg G)$ *is true, if and only if* $\exists x\, (\neg F(x) \vee G)$ *is false, if and only if* $\exists x\, (F(x) \to G)$ *is false.*

The foregoing quite informal argument may be carried out formally in terms of the valuations in a first-order structure; the precise details will depend on precisely how valuations are defined, and will differ from textbook to textbook.

(ii) Show that the following is logically valid in any non-empty domain.

$$(\exists x\, (F(x) \vee G(x)) \leftrightarrow (\exists x\, F(x) \vee \exists x\, G(x))).$$

If $\exists x\, (F(x) \vee G(x))$ *is true in a structure, then there will be some element* a *such that* $(F(a) \vee G(a))$ *is true; then* $F(a)$ *is true or* $G(a)$ *is true, and hence* $\exists x\, F(x)$ *is true or* $\exists x\, G(x)$ *is true, hence* $(\exists x\, F(x) \vee \exists x\, G(x))$ *is true.*

In the opposite direction, if $(\exists x\, F(x) \vee \exists x\, G(x))$ *holds in a structure, then one of* $\exists x\, F(x)$ *and* $\exists x\, G(x)$ *holds; in the first case (the second case is similar), there is an element* a *of the structure such that* $F(a)$ *holds, whence of course* $(F(a) \vee G(a))$ *does too; then* $\exists x\, (F(x) \vee G(x))$ *holds.*

(iii) Show that

$$(\forall x\, (F(x) \vee G(x)) \leftrightarrow (\forall x\, F(x) \vee \forall x\, G(x)))$$

is not logically valid.

In the natural numbers $\mathbb{N}$*, interpret* $F(x)$ *as "*x *is even" and* $G(x)$ *as "*x *is odd".*

(iv) Show that

$$((\forall x\, F(x) \to G) \leftrightarrow \forall x\, (F(x) \to G)),$$

is not logically valid, where x is not free in G.

In the natural numbers, let G *stand for some false statement, such as "*$0 = 1$*", and let* $F(x)$ *stand for "*x *is even". Then* $\forall x\, F(x)$ *is false, so that* $(\forall x\, F(x) \to G)$ *is true; while* $(F(x) \to G)$ *is not always true (if* x *is even it fails), so* $\forall x\, (F(x) \to G)$ *is false.*

2. (i) Using the axiom schemata

$$(F \to (G \to F)) \qquad \text{(A1)}$$

and

$$((F \to (G \to H)) \to ((F \to G) \to (F \to H))) \qquad \text{(A2)}$$

and the rule of inference Modus Ponens, show that for Γ a set of formulae none of which contain any free variables, and for F and G formulae containing no free variables, that if $\Gamma \cup \{F\} \vdash G$, then $\Gamma \vdash (F \to G)$.

The precise result we will prove is: assuming that $\vdash \phi$ for all instances ϕ of schemes (A1) and (A2), if the inference of G from $\Gamma \cup \{F\}$ only uses the rule Modus Ponens, then $\Gamma \vdash (F \to G)$.

This is an induction on the complexity of the proof of G from $\Gamma \cup \{F\}$.

If G is an element of Γ or a logical axiom, then $\Gamma \vdash G$. Hence $\Gamma \vdash (G \to (F \to G))$, since this is an instance of (A1). Hence $\Gamma \vdash (F \to G)$, by Modus Ponens.

If $G = F$, then $(F \to (F \to F))$ and $(F \to ((F \to F) \to F))$ are instances of (A1), and $((F \to ((F \to F) \to F)) \to ((F \to (F \to F)) \to (F \to F)))$ is an instance of (A2). Two applications of Modus Ponens gives us that $\vdash (F \to F)$, from which it easily follows that $\Gamma \vdash (F \to F)$. (If you are familiar with the Thinning Rule, then this step appears to be an application of it. But actually the validity of this step can be proved without an appeal to the Thinning Rule.)

Now suppose that in the proof of G from $\Gamma \cup \{F\}$, G is deduced using Modus Ponens from H and $(H \to G)$.

Then $((F \to (H \to G)) \to ((F \to H) \to (F \to G)))$ is an instance of (A2).

Now using Modus Ponens, and the inductive hypothesis that $\Gamma \vdash (F \to H)$ and $\Gamma \vdash (F \to (H \to G))$, we have that $\Gamma \vdash (F \to G)$.

(ii) Show that $\forall x\, A(x)$ is a logical consequence of $A(x)$ (that is, in any interpretation in which $A(x)$ is true, $\forall x\, A(x)$ is true also), but that $(A(x) \to \forall x\, A(x))$ is not logically valid (that is, there is an interpretation in which it is not true).

This is a technicality concerning the way in which formulae containing free variables are interpreted. We say that $A(x)$ is true in a structure provided that it is true under every possible valuation of the variable letter x, that is, if $A(a)$ is true for every individual a in the structure. This means that $A(x)$ and $\forall x\, A(x)$ are true under precisely the same circumstances. However, in $(A(x) \to \forall x\, A(x))$, the first x is free whereas the remaining ones are bound; so this formula is true provided $(A(a) \to \forall x\, A(x))$ is true for every element a of the structure.

So, in $\mathbb{N}$, let $A(x)$ be "x is equal to zero". Then $(A(x) \to \forall x\, A(x))$ is false when the free occurrence of x is thought of as referring to 0.

(iii) Show that if F is a sentence (that is, a formula with no free variables) and $\mathfrak{M}$ is a structure, then either $\mathfrak{M} \models F$ or $\mathfrak{M} \models \neg F$. Give an example of a formula $F(x)$ with a free variable, and a structure $\mathfrak{M}$, such that it is not true either that $\mathfrak{M} \models F(x)$ or that $\mathfrak{M} \models \neg F(x)$.

Let $\mathfrak{M}$ be $\mathbb{N}$, and let $F(x)$ express "x is even".

Variable letters in formal languages are a little like pronouns in natural languages. We might say that the pronoun "its" is bound in the sentence "Every unhappy family is unhappy in its own way", and we can say whether we consider the sentence to be true or false; but "it" is free in "it was written by Tolstoy", where we cannot say whether the sentence is true or false, because no definite meaning has been assigned to "it" (is it Anna Karenina*, or* The Brothers Karamazov*, or my shopping list?).*

3. (i) Write down a set of axioms and rules of inference for first order predicate calculus.

There are many, many ways of doing this; I will give one in the lectures in due course.

They each have (i) a (possibly empty) set of logical axioms*, and (ii) a set of* rules of inference *which enable one to build more complicated proofs out of simple ones.*

(ii) State the Completeness Theorem for your system, and sketch a proof of it.

There are several ways to state this. One is: "If Γ is a set of sentences, and ϕ is a sentence, then $\Gamma \vdash \phi$ (that is, ϕ may be proved from Γ) if and only if $\Gamma \vDash \phi$ (that is, in any structure in which Γ is true, ϕ is true as well).

An informal version of the Completeness Theorem is, "ϕ may be proved from Γ if and only if ϕ is true whenever Γ is".

But note the condition "in any structure". Some of the structures we may find ourselves looking at may be quite strange.

The Completeness Theorem has two parts. The first, the Soundness Theorem, is the statement that if $\Gamma \vdash \phi$, then $\Gamma \vDash \phi$, or informally, anything you can prove is always true. This is done by induction on the complexity of the deduction of ϕ from Γ, and boils down to checking that all logical axioms are always true, and that all rules of inference are sound.

The second part, the Adequacy Theorem, states the converse: that if $\Gamma \vDash \phi$, then $\Gamma \vdash \phi$. This is now generally done by an argument due to Henkin, which goes thus.

Add to the language a constant symbol c_ϕ for every formula $\phi(x)$ with just one free variable x. (This will need to done in an infinite recursion, since we want to do this even if the formula $\phi(x)$ contains some of these new constant symbols.) Let Δ be the set of all formulae $(\exists x\, \phi(x) \to \phi(c_\phi))$.

Suppose that $\Gamma \nvdash \phi$. Then $\Gamma \cup \{\neg\phi\}$ is consistent, and in fact so is $\Gamma \cup \Delta \cup \{\neg\phi\}$.

Extend to a complete consistent set Σ.

Let M be the set of all the constant symbols c_ϕ, and let $\mathfrak{M}$ be the set of equivalence classes on M under the equivalence relation $\sim$ defined so that $c_\phi \sim c_\psi$ iff $(c_\phi = c_\psi) \in \Sigma$, turned into an L-structure in the obvious way (consulting the formulae in Σ). Then $\mathfrak{M} \vDash \Sigma$.

It follows at once that $\Gamma \cup \{\neg\phi\}$ is satisfiable, so $\Gamma \nvDash \phi$.

You may have only seen this argument for when L is a countable language. But it works for languages of any size, assuming the Axiom of Choice.

(iii) State the Compactness Theorem, and deduce it from the Completeness Theorem.

The Compactness Theorem states: "A set of formulae Γ is satisfiable (that is, there is a structure and a valuation making all formulae in Γ simultaneously true) if and only if every finite subset of Γ is satisfiable".

This bizarre theorem may be deduced from the Completeness Theorem very quickly: Γ is unsatisfiable if and only if $\Gamma \vDash (\phi \wedge \neg\phi)$ if and only if $\Gamma \vdash (\phi \wedge \neg\phi)$. But any deduction of $(\phi \wedge \neg\phi)$ from Γ mentions only a finite set Γ_0 of elements of Γ. So $\Gamma_0 \vdash (\phi \wedge \neg\phi)$, so $\Gamma_0 \vDash (\phi \wedge \neg\phi)$, so Γ_0 is unsatisfiable.

(iv) Use the Compactness Theorem, or some other method, to construct a countably infinite structure $\mathfrak{N}$ of which the natural numbers $\mathbb{N}$ (equipped with a constant symbol to refer to 0, a unary function to refer to the function $n \mapsto n+1$, and binary functions to refer to additional and multiplication) is a proper subset and an elementary substructure.

Add a constant symbol c to the language we are using, let Θ be the set of all sentences true in $\mathbb{N}$ (which will allow us to completely reconstruct $\mathbb{N}$, since the language contains *terms referring to all elements of* $\mathbb{N}$ *and so* Θ *contains the complete addition and multiplication tables of* $\mathbb{N}$*), and let* Σ *be the set of sentences* $\{\neg(c = \overline{n}) : n \in \mathbb{N}\}$*, where* $\overline{n}$ *is some term referring to* n*. Every finite subset of* $\Theta \cup \Sigma$ *is satisfiable (in* $\mathbb{N}$*, with* c *referring to some large enough natural number), so by the Compactness Theorem* $\Theta \cup \Sigma$ *has some model* $\mathfrak{N}$*. Because* Θ *is true in* $\mathfrak{N}$*, and* Θ *includes the addition and multiplication tables of* $\mathbb{N}$*,* $\mathfrak{N}$ *contains a substructure isomorphic to* $\mathbb{N}$*. This is an elementary substructure as well; but the interpretation of* c *in* $\mathfrak{N}$ *is not in this copy of* $\mathbb{N}$*, so the copy of* $\mathbb{N}$ *in* $\mathfrak{N}$ *is a proper elementary substructure.*

Deduce that the theory of $\mathbb{N}$ is not $\aleph_0$-categorical, that is, that it has distinct, non-isomorphic countable models.

Trivial.

(v) (Optional, and fiddly, but worth knowing): Let $\mathfrak{M}$ be an infinite model of some theory T in a countable language L of first order predicate calculus. Show that there is a countably infinite substructure $\mathfrak{N}$ of $\mathfrak{M}$ such that $\mathfrak{N}$ is an elementary substructure of $\mathfrak{M}$. Deduce the *Countable Downward Löwenheim-Skolem Theorem*: that if T has an infinite model, then it has a countable model.

Without loss of generality (expanding the language if necessary), suppose L *and* T *to contain* Skolem functions*: that is, for each formula* $\phi(x, y_1, \ldots, y_n)$ *of* L *with just the free variables* x*,* y_1*, …,* y_n*, suppose* L *to contain a function symbol* f_ϕ *and* T *to contain the sentence* $\forall y_1, \ldots, y_n\, (\exists x\, \phi(x, y_1, \ldots, y_n) \to \phi(f_\phi(y_1, \ldots, y_n), y_1, \ldots, y_n))$*. Let* $\mathfrak{N}$ *be the smallest subset of* $\mathfrak{M}$ *closed under all the Skolem functions.*

Note that the "without loss of generality" bit of the argument involves choosing interpretations for an infinite number of symbols f_ϕ*, and this involves a use of the Axiom of Choice. This is bound to be necessary since, for one thing, without the Axiom of Choice, an infinite set need not have any countably infinite subsets at all.*

(vi) (Optional, and only for those who know enough set theory): Assume that T is a theory in a countable language L of first order predicate calculus, which has arbitrarily large infinite models. Show that T has a model of every infinite size. (Your answer to the previous part will have assumed the Axiom of Choice in some form. Your answer to this one is likely to use it much more heavily.)

Similar to the above. Suppose L *and* T *to be equipped with Skolem functions.*

Let κ *be an infinite cardinal, and let* $\mathfrak{M}$ *be a model of* T *of size greater than or equal to* κ*. Let* A *be some subset of* $\mathfrak{M}$ *of size* κ*. Let* $\mathfrak{N}$ *be the smallest subset of* $\mathfrak{M}$ *closed under all Skolem functions, such that* $A \subseteq \mathfrak{N}$*.*

Then $\mathfrak{N}$ *is an elementary substructure of* $\mathfrak{M}$ *and thus a model of* T*, and has size* κ*.*

(vii) (Definitely optional, and requires set theory): Prove that a theory T in a countable first order language that has an infinite model has arbitrarily large infinite models.

Let κ *be an infinite cardinal, and add to the language* L *a set* C *of size* κ *of constant symbols, and consider the set* $T \cup \{\neg(c = d) : c, d \in C, c \neq d\}$*. Every finite subset of this is satisfiable because* T *has an infinite model; thus by the Compactness Theorem the whole set is; and the model in which all these formulae are true, must have size greater than or equal to* κ*.*

This may also be done economically using ultraproducts (if you are familiar with them).

(viii) (Optional: requires set theory) Deduce the *Löwenheim-Skolem Theorem*: if a theory T in a countable first order language has an infinite model, then it has a model of every infinite cardinality.

Now easy.

4. (Optional: uses set theory and model theory) (i) Let $\mathfrak{N}$ be a countable model of the theory of $\mathbb{N}$ which is not isomorphic to $\mathbb{N}$. Prove that $\mathfrak{N}$ is totally ordered, and that it has an initial segment isomorphic to $\mathbb{N}$.

The theory of $\mathbb{N}$ *contains the axioms of a total order, and also contains the sentences* $\overline{n} \leq \overline{n+1}$ *for each* n*, where* $\overline{n}$ *is some term referring to* n*, and also the sentences*

$$\forall x\,(x = \overline{0} \vee x = \overline{1} \vee \cdots \vee x = \overline{n} \vee x \geq \overline{n+1})$$

for each n*.*

Note that the theory of $\mathbb{N}$ *does NOT contain the axioms of a well-ordering, since these cannot be expressed in a first order language. Hence* $\mathfrak{N}$ *will typically not be well-ordered.*

(ii) Show that every non-standard element of $\mathfrak{N}$ belongs to an interval which is order-isomorphic to $\mathbb{Z}$.

The theory of $\mathbb{N}$ *states that every non-zero element of* $\mathbb{N}$ *has an immediate predecessor and an immediate successor. The result follows at once.*

Note the corollary: if $\mathfrak{N}$ *is not isomorphic to* $\mathbb{N}$*, then it is definitely not well-ordered.*

(iii) Show that there is no final such interval order-isomorphic to $\mathbb{Z}$, no initial one, and that between any two such intervals, there is another.

If n *is a non-standard element of* $\mathfrak{N}$*, then, since multiplication is defined in the theory of* $\mathbb{N}$*, there must be an element of* $\mathfrak{N}$ *which* $\mathfrak{N}$ *believes to be the product* $2n$ *of* 2 *with* n*. Since in* $\mathbb{N}$*, for all non-zero* m*,* $m < 2m$*, it follows that in* $\mathfrak{N}$*,* $n < 2n$*. And* n *and* $2n$ *must belong to different copies of* $\mathbb{Z}$*, or else there would exist a standard natural number* m *such that* $\mathfrak{N} \models 2n = n + m$*, and thus (since the usual laws of arithmetic on* $\mathbb{N}$ *are contained in its theory and are thus true in* $\mathfrak{N}$*)* $\mathfrak{N} \models n = m$*, contradicting the assumption that* n *was non-standard.*

Thus there is no final copy of $\mathbb{Z}$ *in* $\mathfrak{N}$*.*

The fact that there is no initial copy of $\mathbb{Z}$ *uses a similar argument with either* $\frac{1}{2}n$ *or* $\frac{1}{2}(n-1)$*, depending on which of* n *and* $n-1$ $\mathfrak{N}$ *believes to be even.*

If m *and* n *are non-standard natural numbers belonging to different copies of* $\mathbb{Z}$*, then we find another copy of* $\mathbb{Z}$ *in between by considering either* $\frac{1}{2}(m+n)$ *or* $\frac{1}{2}(m+n-1)$*, depending on which of* $m+n$ *and* $m+n-1$ $\mathfrak{N}$ *believes to be even.*

(iv) Deduce that $\mathfrak{N}$ is order-isomorphic to $\mathbb{N} \oplus (\mathbb{Q} \otimes \mathbb{Z})$, where the operators $\oplus$ and $\otimes$ have the following meanings. If $(P, \leq_P)$ and $(Q, \leq_Q)$ are total orders, with P and Q being disjoint sets, then $P \oplus Q$ is the set $P \cup Q$ equipped with the order $\leq_{P \oplus Q}$ in which Q comes after P; that is, $x \leq y$ iff $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$, or $x \in P$ and $y \in Q$; and $P \otimes Q$ is the cartesian product $P \times Q$ equipped with the lexigraphic order, in which $(p, q) \leq (p', q')$ iff either $p < p'$, or $p = p'$ and $q \leq q'$.

This follows from the theorem that every countable dense linear order without end-points is order-isomorphic to $\mathbb{Q}$*.*

Deduce that all countable models of the theory of $\mathbb{N}$ which are not isomorphic to $\mathbb{N}$, are order-isomorphic to each other.

Now trivial.

Note that we have NOT proved that such models of the theory of $\mathbb{N}$ *are isomorphic, considered as models of the theory of* $\mathbb{N}$*; we have proved a much weaker statement.*

(v) Let $A \subseteq \mathbb{N}$ be the set of all primes. If $f : A \to \{0,1\}$, let Σ_f be the following set of formulae of an appropriate first-order language of arithmetic, with an additional constant symbol c:

$$\Sigma_f = \{(\overline{p} \mid c) : f(p) = 1\} \cup \{(\overline{p} \not\mid c) : f(p) = 0\},$$

where $\overline{p}$ is some term referring to p, and $m \mid n$ means "m is a factor of n".

Prove that there is a countable model $\mathfrak{M}_f$ of the theory of $\mathbb{N}$ in which Σ_f is true.

This is an application of the Compactness Theorem.

Strictly speaking, because we have added a constant symbol c, we are now using a different language.

In what follows, I will sometimes commit the abuse of thinking of $\mathfrak{M}_f$ as a model of the language of the natural numbers (without the constant symbol c).

(vi) Deduce that there are $2^{\aleph_0}$ different non-isomorphic countable models of the theory of $\mathbb{N}$.

It is a nuisance that (committing the abuse mentioned above, and forgetting the constant symbol c) some of the models $\mathfrak{M}_f$ may be isomorphic to each other; otherwise we could just say "there are $2^{\aleph_0}$ different f, so there are $2^{\aleph_0}$ different $\mathfrak{M}_f$".

We have to be a little bit careful.

So, suppose that κ is an infinite cardinal, and $\kappa \leq 2^{\aleph_0}$. Suppose that there are exactly κ countable models of the theory of $\mathbb{N}$*. List them all as $\{\mathfrak{M}^\alpha : \alpha < \kappa\}$.*

Now, for each $\alpha < \kappa$, let A_α be the set of all f such that $\mathfrak{M}^\alpha$ is isomorphic to $\mathfrak{M}_f$. The fact that $\mathfrak{M}^\alpha$ is countable means that A_α is countable. For if not, there must exist $f \neq g$ such that $\mathfrak{M}_f$ and $\mathfrak{M}_g$ are isomorphic, via an isomorphism π such that π sends the interpretation of the new constant c in $\mathfrak{M}_f$, to the interpretation of c in $\mathfrak{M}_g$. But Σ_f and Σ_g are inconsistent with each other, so this is impossible.

Now every f must belong to some A_α.

Hence $|\bigcup_{\alpha<\kappa} A_\alpha| \geq 2^{\aleph_0}$.

This is only possible if $\kappa = 2^{\aleph_0}$, as required.