Gödel Incompleteness Theorems: Problem sheet 1

1. (Optional: have a go at this if you've not seen PA before.) Show that all of the following can be proved from PA.

(i) Every natural number is either even or odd (i.e. for all n, either there exists m such that $n = 2.m$, or there exists m such that $n = (2.m)^+$).

(ii) Addition is associative.

(iii) Addition is commutative. (Hard.)

(iv) Multiplication is associative.

(v) Multiplication is commutative. (Harder.)

(vi) Multiplication is distributive over addition.

2. Describe informally a method by which it can be decided whether an expression of $\mathscr{L}_E$ is a term, a formula, or neither.

3. (i) Write down a true sentence in $\mathscr{L}_E$ containing exactly eight symbols, and write down its Gödel number according to the system given in lectures (write it in base 13 if you prefer).

(ii) Write down a true sentence in the language $\mathscr{L}_E$ containing $\neg$, $\rightarrow$ and $\forall$ that is not logically valid (ie. that is not true in every logical structure whatever), and give an informal argument to show that it is true.

4. (i) Show that the relation "x divides y" can be expressed in $\mathscr{L}_E$.

(ii) Show that the property of being a power of 7 can be expressed in $\mathscr{L}_E$. Can it be expressed without using exponentiation?

(iii) Show that if A is a set and g is a (unary) function, and both A and g are definable in $\mathscr{L}_E$, then $g^{-1}(A)$ is also definable in $\mathscr{L}_E$.

5. (i) Show that for any formula $F(v_i, v_j)$,

$$\mathrm{PA} \vdash (\exists v_j\, \exists v_i\, F(v_i, v_j) \leftrightarrow \exists v_k\, (\exists v_j \leq v_k)(\exists v_i \leq v_k) F(v_i, v_j)).$$

(ii) Show that for any formula $F(v_i, v_j)$,

$$\mathrm{PA} \vdash ((\forall v_j \leq v_k) \exists v_i\, F(v_i, v_j) \leftrightarrow \exists v_r (\forall v_j \leq v_k)(\exists v_i \leq v_r) F(v_i, v_j)).$$

6. (i) Show that the function

$$p(m, n) = \tfrac{1}{2}(m + n + 1)(m + n) + m$$

is a pairing function on the natural numbers, that is, it is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$; and show that it is Σ_0 (that is, the statement "$k = [m, n]$" is provably Σ_0).

(ii) Show that there are two one-place Σ_0-functions p_l and p_r such that $p_l(p(m, n)) = m$ and $p_r(p(m, n)) = n$.

7. Show that

(i) for $n > 0$, formulae provably Σ_n with respect to PA are closed under existential quantification, and formulae provably Π_n with respect to PA are closed under universal quantification,

(ii) formulae provably equivalent Σ_n with respect to PA are closed under conjunction and disjunction, and formulae provably Π_n with respect to PA are closed under conjunction and disjunction,

(iii) formulae that are provably Δ_n with respect to PA are closed under conjunction and disjunction.