Gödel Incompleteness Theorems: Problem sheet 2

1. Show that the set $\{n : \mathrm{PAE} \vdash E_n[\overline{n}]\}$ is expressible in complexity Σ_1.

2. Show that for any two formulae $F(v_1)$ and $G(v_1)$ in $\mathscr{L}_E$ with one free variable, there exist sentences X and Y such that the sentences $(X \leftrightarrow G(\overline{\ulcorner Y \urcorner}))$ and $(Y \leftrightarrow F(\overline{\ulcorner X \urcorner}))$ are both true.

3. Show that if S is a definable set of sentences in $\mathscr{L}_E$, and Pr_S is an associated proof predicate, and X and Y are any formulae, then

$$\mathrm{PAE} \vdash (\mathrm{Pr}_S(\overline{\ulcorner X \to Y \urcorner}) \to (\mathrm{Pr}_S(\overline{\ulcorner X \urcorner}) \to \mathrm{Pr}_S(\overline{\ulcorner Y \urcorner}))).$$

4. Show that the following functions are primitive recursive.

(i) $P(n)$, which is $n-1$ if $n > 0$ and 0 if $n = 0$.

(ii) $S(m, n)$, which is $m - n$ if $m \geq n$, and 0 if $m < n$.

(iii) $M(m, n) = m.n$.

(iv) $E(m, n) = m^n$.

(v) $L(m, n) = \min(m, n)$, and $U(m, n) = \max(m, n)$.

(vi) $G(n) = \min_{m \leq n} F(m)$ and $H(n) = \max_{m \leq n} F(m)$, where F is primitive recursive.

5. (i) Show that every true, quantifier-free sentence is provable from PAE.

(ii) Prove that if ϕ is quantifier-free, and $\exists v_i \leq \overline{n}\, \phi$ is a sentence, then there is a quantifier-free sentence ϕ' which is true if and only if ϕ is true. [*Note that n here is a fixed natural number, and the choice of ϕ' will depend on the choice of n.*]

(iii) Prove that every true Σ_0 sentence is provable from PAE.

(iv) Deduce that every true Σ_1 sentence is provable from PAE.

6. Let $F(\overline{n})$ be the statement "there exists a Σ_1 formula ϕ such that $n = \ulcorner \phi \urcorner$". [*Assume that this is expressible in complexity Σ_0.*]

If ϕ is any formula, and n and k are natural numbers, write $\phi(\overline{n}, \overline{k}, \mathbf{0})$ for the result of substituting $\overline{n}$ for all free occurrences of v_1 in ϕ, $\overline{k}$ for all free occurrences of v_2, and $\overline{0}$ for all other free variables. [*Assume that the statement $G(\overline{m}, \overline{m'}, \overline{n}, \overline{k})$ which we define as "If ϕ is such that $m = \ulcorner \phi \urcorner$, then $m' = \ulcorner \phi(\overline{n}, \overline{k}, \mathbf{0}) \urcorner$" can be expressed in Σ_0.*]

(i) Show that the statement $H(\overline{m}, \overline{n}, \overline{k})$, which we define as "$F(\overline{m})$ is true, and if ϕ satisfies $m = \ulcorner \phi \urcorner$, then $\phi(\overline{n}, \overline{k}, \mathbf{0})$" is expressible in complexity Σ_1.

(ii) Prove that the statement $K(\overline{m}, \overline{n})$ which we define as "$F(\overline{m})$ is true, and if ϕ is such that $m = \ulcorner \phi \urcorner$, then there exists k such that $\phi(\overline{n}, k, \mathbf{0})$" is expressible in complexity Σ_1.

(iii) Show that $\neg K(\overline{n}, \overline{n})$ is not expressible in complexity Σ_1.

(iv) (Optional: hard) Let Γ be the smallest set of partial functions with the following properties.

(α) Every recursive partial function belongs to Γ.

(β) The characteristic function of the set $\{(m, n) : K(\overline{m}, \overline{n}) \text{ is false}\}$ belongs to Γ.

(γ) Γ is closed under substitution, primitive recursion, and minimalisation.

Here the minimalisation operator, as applied to partial functions f, is defined as follows. g is defined from minimalisation from f iff, for all n, $g(n_1, \ldots, n_k, n)$ is the least m such that for all $l \leq m$, $f(n_1, \ldots, n_k, l)$ is defined, and such that $f(n_1, \ldots, n_k, m) = 0$, if such an m exists; otherwise $g(n_1, \ldots, n_k, n)$ is undefined.

Sketch an argument that the elements of Γ are precisely the partial functions that can be defined in complexity Σ_2.