Gödel Incompleteness Theorems: Problem sheet 3

1. Show that if a system S is Σ_0-complete and ω-consistent, then it is Σ_2-sound.

2. Show that consistency is strictly weaker than 1-consistency.

3. (i) Show how to construct a sentence, using the Diagonal Lemma, that "says", "this sentence, when added to PA, results in a system that is ω-inconsistent".

Hint: We will use a notation (due to Feferman) of putting a dot over a free variable in a formula that occurs within $\ulcorner\urcorner$ *to signify that the Gödel number is of a formula that varies with the values of that variable, rather than being the Gödel number of the formula in which that variable occurs as a part (for example,* $\forall v'\ \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner\neg 0 = \dot{v}'^{+}\urcorner})$ *says that for each number* n, $\mathrm{Pr}_{\mathrm{PA}}\overline{\ulcorner\neg 0 = \overline{n}^{+}\urcorner}$ *holds, while* $\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner\neg = v'^{+}\urcorner})$, *with no dot, says that a particular formula* $\neg v'^{+}$ *is provable, and says it by referring to its Gödel number; since* $\overline{\ulcorner\neg v'^{+}\urcorner}$ *is just a number,* $\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner\neg = v'^{+}\urcorner})$ *does not contain* v' *as a free variable, and so* $\forall v'\ \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner\neg 0 = v'^{+}\urcorner})$ *(still with no dot) has exactly the same meaning).*

With this notation, show that there is a formula of $\mathscr{L}$ *which expresses the following, where* $m\ P\ n$ *is the relation "*m *is a part of* n *in base* 13*":*

$$\exists v_2\,((\overline{\ulcorner v_4\urcorner}\ P\ v_2)\wedge\neg(\overline{\ulcorner\forall v_4\urcorner}\ P\ v_2)\wedge\mathrm{Pr}(\overline{\ulcorner(E_{\dot{v}_1}\to\exists v_4 E_{\dot{v}_2})\urcorner})\wedge\forall v_3\ \mathrm{Pr}(\overline{\ulcorner(E_{\dot{v}_1}\to\neg E_{\dot{v}_2}[\overline{\ulcorner\dot{v}_3\urcorner}])\urcorner})).$$

Note that v_1 *is free in this expression.*

(ii) Prove that the result in the last problem but one is the best possible, in the sense that there exists a system S that is ω-consistent and which proves a false Σ_3-sentence. (Assume that PA is true in $\mathbb{N}$.)

4. (i) Show that every finite subset of the axioms of R has a finite model.
Hint: consider modular arithmetic.

(ii) Show that R is not finitely axiomatisable.

(iii) Show that Q is a proper extension of R.

(iv) Show that PA is a proper extension of Q.

5. (i) Show that if a theory S is ω-consistent, then at least one of $S\cup\{X\}$ and $S\cup\{\neg X\}$ is ω-consistent.

(ii) Show that there is one and only one complete ω-consistent extension of PA. Take as given that PA is sound.

(This extension will be the theory of $\mathbb{N}$, which we know not to be definable.)

(iii) Explain why the following complete extension S of PA is not ω-consistent. Let $\{X_n : n\in\mathbb{N}\}$ be a listing of all sentences of $\mathscr{L}$. Let K be a sentence such that K is false and $\mathrm{PA}\cup\{K\}$ is ω-consistent, and let S_0 be $\mathrm{PA}\cup\{K\}$. Let S_{n+1} be $S_n\cup\{X_n\}$ if $S_n\cup\{X_n\}$ is ω-consistent, otherwise let S_{n+1} be $S_n\cup\{\neg X_n\}$. For each i, S_i is ω-consistent by part (i). Let $S=\bigcup_{n\in\mathbb{N}} S_n$.