Gödel Incompleteness Theorems: Problem sheet 4

1. (i) Prove that for any sentence X, $\mathrm{PA} \vdash (\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner (\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X \urcorner}) \to X) \urcorner}) \to \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X \urcorner}))$.

[*Note that this is not exactly the same as Löb's Theorem. You can use Löb's Theorem however. You may find it useful to try to prove that* $\mathrm{PA} \vdash (\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner L \urcorner}) \to L)$, *where* $L = (\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner (\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X \urcorner}) \to X) \urcorner}) \to \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X \urcorner}))$.]

(ii) Show that $\mathrm{PA} \vdash (\mathrm{Con}_{\mathrm{PA}} \to \neg \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \mathrm{Con}_{\mathrm{PA}} \urcorner}))$.

(iii) Show that for X any Π_1 sentence, if $\mathrm{PA} \cup \{\neg \mathrm{Con}_{\mathrm{PA}}\} \vdash X$, then $\mathrm{PA} \vdash X$.

2. Show that $\mathrm{PA} \vdash (\mathrm{Con}_{\mathrm{PA}} \to \mathrm{Con}_{\mathrm{PA} \cup \neg \mathrm{Con}_{\mathrm{PA}}})$.

3. Verify that the following formulae are fixed points for the operators $p \mapsto A(p)$ given. (These are not in the exact form you would derive by working through the proof of Theorem 7.2.1., but the uniqueness in that theorem is only up to provable logical equivalence.)

(i) $(\Box\, q \to q)$ is a fixed point for $A(p) = (\Box\, p \to q)$.

(ii) $\Box\, q$ is a fixed point for $A(p) = \Box(p \leftrightarrow (\Box\, p \to q))$.

(iii) $\Box(\Box\, q \wedge \Box\, r)$ is a fixed point for $A(p) = \Box(\Box(p \wedge q) \wedge \Box(p \wedge r))$.

4. Find fixed points for

(i) $A(p) = (\Box\, p \to \Box \neg p)$,

(ii) $A(p) = (\Box\, p \wedge \neg \Box \neg p)$.