ANALYSIS I

Axioms for the Real Numbers

Algebraic Properties

For every pair of real numbers $a, b \in \mathbb{R}$ there is a unique real number $a + b$, called their 'sum'.
For every pair of real numbers $a, b \in \mathbb{R}$ there is a unique real number $a \cdot b$, called their 'product'.
For each real number $a \in \mathbb{R}$ there is a unique real number $-a$, called its 'negative' or 'additive inverse'.
For each real number $a \in \mathbb{R}$, with $a \neq 0$, there is a unique real number $\frac{1}{a}$, called its 'reciprocal' or 'multiplicative inverse'.
There is a special element $0 \in \mathbb{R}$ called 'zero' or 'the additive identity'.
There is a special element $1 \in \mathbb{R}$ called 'one' or 'the multiplicative identity'.

The following hold for all real numbers a, b, c:

A1 $a + b = b + a$ [+ is commutative]
A2 $a + (b + c) = (a + b) + c$ [+ is associative]
A3 $a + 0 = a$ [zero and addition]
A4 $a + (-a) = 0$ [negatives and addition]

M1 $a \cdot b = b \cdot a$ [$\cdot$ is commutative]
M2 $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ [$\cdot$ is associative]
M3 $a \cdot 1 = a$ [the unit element and multiplication]
M4 If $a \neq 0$ then $a \cdot \frac{1}{a} = 1$ [reciprocals and multiplication]

D $a \cdot (b + c) = a \cdot b + a \cdot c$ [$\cdot$ distributes over +]

Z $0 \neq 1$ [to avoid total collapse]

Notation: we write
$$\begin{cases} ab & \text{for } a \cdot b \\ a - b & \text{for } a + (-b); \\ a/b & \text{for } a\frac{1}{b} \quad (b \neq 0); \\ a^{-1} & \text{for } \frac{1}{a} \quad (a \neq 0). \end{cases}$$

Order Properties.

There exists a subset $\mathbb{P}$ of $\mathbb{R}$ called the '(strictly) positive numbers' such that for all a, b in $\mathbb{R}$
P1 If $a \in \mathbb{P}$ and $b \in \mathbb{P}$ then $a + b \in \mathbb{P}$. [addition and the order]
P2 If $a \in \mathbb{P}$ and $b \in \mathbb{P}$ then $a \cdot b \in \mathbb{P}$. [multiplication and the order]
P3 Exactly one of $a \in \mathbb{P}$, $a = 0$, $-a \in \mathbb{P}$ is true [trichotomy]

Notation: we write
$$\begin{cases} a > b & \text{for } a - b \in \mathbb{P}; \\ a < b & \text{for } b - a \in \mathbb{P}; \\ a \geqslant b & \text{for } a - b \in \mathbb{P} \text{ or } a = b; \\ a \leqslant b & \text{for } b - a \in \mathbb{P} \text{ or } b = a. \end{cases}$$

Completeness Property

Upper bound: Suppose that $E \subseteq \mathbb{R}$, and that $b \in \mathbb{R}$ is such that $x \leqslant b$ for all $x \in E$. We then say that 'b is an upper bound of E', and that 'E is bounded above.' Notation: we shall write $E^{\uparrow}$ to denote the set of upper bounds of E.

Supremum: Suppose that E is a non-empty subset of $\mathbb{R}$ which is bounded above. Assume that $s \in \mathbb{R}$ is such that

(a) $s \in E^{\uparrow}$ [s is *an* upper bound of E]

(b) $b \in E^{\uparrow}$ implies $s \leqslant b$ [s is the *least* upper bound of E]

Then s is called the *supremum* of E (notation: $s = \sup E$).

The Completeness Axiom

Let E be a non-empty subset of $\mathbb{R}$ which is bounded above. Then $\sup E$ exists. [completeness]