Foundations of Stochastic Analysis

Exercise Sheet

1. An adapted stochastic process $(X_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ taking values in $\mathbb{R}^d$ is called a *Lévy process*, if for any t, $X_s \to X_t$ in probability as $s \to t$, and for any $t > s$ the increment $X_t - X_s$ is independent of $\mathcal{F}_s$.

   (a) Show that for any $0 \leq t_0 < t_1 < \cdots < t_n$ random variables

   $$X_{t_0}\ ,\ \ X_{t_1} - X_{t_0}\ ,\ \ \cdots\ ,\ X_{t_n} - X_{t_{n-1}}$$

   are mutually independent. That is, (X_t) possesses *independent increments*.

   (b) Let $\varphi_{s,t}$ be the characteristic function of $X_t - X_s$ (for $t \geq s \geq 0$):

   $$\varphi_{s,t}(\xi) = \mathbb{E}\left\{\exp\left(i\langle \xi, X_t - X_s\rangle\right)\right\} \quad \text{for } \xi \in \mathbb{R}^d.$$

   Show that, as a function of (s,t,ξ), $\varphi_{s,t}(\xi)$ is continuous. For any ξ, $\varphi_{t,t}(\xi) = 1$, and

   $$\varphi_{s,t}(\xi)\varphi_{t,u}(\xi) = \varphi_{s,u}(\xi),$$

   (c) Prove that for any $\xi \in \mathbb{R}^d$ and $t \geq s$, $\varphi_{t,s}(\xi) \neq 0$.
   Hint: Set $t_0 = \inf\{t \geq s : \varphi_{t,s}(\xi) = 0\}$ and show that $t_0 = +\infty$ by contradiction.

   (d) For $\xi \in \mathbb{R}^d$, define

   $$Z_t = \frac{1}{\varphi_{0,t}(\xi)} \exp\left(i\langle \xi, X_t - X_0\rangle\right) \qquad \text{for } t \geq 0.$$

   Show that $Z = (Z_t)_{t\geq 0}$ is a martingale.

2. . Let $\{B_t : t \geq 0\}$ be a standard Brownian motion in $\mathbb{R}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Fix a constant $C > 0$, and define

   $$A_n = \left\{\omega \in \Omega\colon\ \ \exists s \in [0,1] \text{ so that } |B_t(\omega) - B_s(\omega)| \leq C|t-s| \text{ if } |t-s| \leq \frac{3}{n}\right\};$$

   define

   $$Y_{k,n} = \max\left\{\left|B_{\frac{k+j}{n}} - B_{\frac{k+j-1}{n}}\right| : j = 0,1,2\right\}\ ,\quad k = 1,\cdots,n-2,$$

   and set

   $$K_n = \left\{\omega \in \Omega\colon\ \text{ at least one } Y_{k,n} \leq \frac{5C}{n}\right\}.$$

   (a) Explain why $Y_{k,n}$ are random variables, and why the sets A_n, K_n are measurable.

   (b) Prove that $n \to A_n$ is increasing, and $A_n \subset K_n$ for any $n \geq 3$.

   (c) Show that, for every $n \geq 3$,

   $$\mathbb{P}\left\{K_n\right\} \leq n\left(\mathbb{P}\left\{|B_{\frac{1}{n}}| \leq \frac{5C}{n}\right\}\right)^3.$$

(d) Show that

$$\mathbb{P}\left\{|B_{\frac{1}{n}}| \leq \frac{5C}{n}\right\} \leq \frac{1}{\sqrt{2\pi}} \frac{10C}{\sqrt{n}}.$$

(e) Prove that $\mathbb{P}\{A_n\} = 0$ for every n, hence conclude that, with probability one, Brownian motion paths are not Lipschitz continuous (and hence not differentiable) at any point.

3. Let $D_n = \{0 = t_0, t_1, t_2, \ldots, t_n = t\}$ denote a deterministic partition of the time interval $[0, t]$. A monotone sequence of partitions is one in which $D_n \subset D_{n+1}$. The p-variation of a stochastic process X up to time t along the partition D_n is given by

$$V^p_{D_n}(X)_t = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^p.$$

(a) Recall that for a Brownian motion B, the quadratic variation $V^2_{D_n}(B)_t \to t$ in L^2 for any monotone sequence of partitions with $m(D_n) = \max\{|t_i - t_{i-1}|; t_i \in D_n\} \to 0$. Show that the convergence is almost sure if $m(D_n)$ goes to 0 sufficiently fast, that is when $\sum_{n=1}^{\infty} m(D_n) < \infty$.

(b) Show that $X_n = V^2_{D_n}(B)_t, n \in \mathbb{N}$ is a reverse time discrete martingale (that is $\mathbb{E}(X_{n-1}|\mathcal{G}_n) = X_n$) with respect to the filtration $\mathcal{G}_n = \sigma(X_m : m \geq n)$. Now show that in fact, for *any* monotone sequence of partitions, the convergence is almost sure by using the martingale convergence theorem for this reverse martingale.

(c) A real-valued process is centred Gaussian, if its finite-dimensional distributions are normal distributions with mean zero. A centred Gaussian process $X = (X_t)_{t\geq 0}$ is called a fractional Brownian motion (FBM) with Hurst parameter $h \in (0, 1)$ if $\mathbb{P}(X_0 = 0) = 1$ and its co-variance function

$$\mathbb{E}(X_t X_s) = \frac{1}{2}\left(t^{2h} + s^{2h} - |t - s|^{2h}\right).$$

i. Show that for any $t > s$, and $p > 0$

$$\mathbb{E}|X_t - X_s|^p = C_p|t - s|^{hp}$$

for some constant C_p depending only on p.

ii. Show that the fractional Brownian motion X has a continuous modification and determine its Hölder exponent.

iii. Determine for which p it has finite p-variation along a monotone sequence of partitions.

4. (a) Let $M_t = \max_{s\leq t} W_s$ be the maximum of Brownian motion up to time t. Using the reflection principle show that the joint distribution of the maximum and the position of the Brownian motion at time t is given by

$$\mathbb{P}(M_t \geq y, W_t \leq x) = \int_{2y-x}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-u^2/2t} du,$$

for $y \geq 0$ and $x \leq y$. Hence write down the density $\mathbb{P}(M_t \geq y, W_t \in dx)$ for $y \geq 0, x \leq y$.

(b) Consider $W_t^\mu = W_t + \mu t$, the Brownian motion with constant drift μ, and write $M_t^\mu = \max_{s \leq t} W_s^\mu$ for its maximum process. Using an appropriate change of measure show that, for $y \geq 0, x \leq y$,

$$\mathbb{P}(M_t^\mu \geq y, W_t^\mu \in dx) = \frac{1}{\sqrt{2\pi t}} e^{\mu x - \frac{1}{2}\mu^2 t - (2y-x)^2/2t} dx.$$

(c) Write down the corresponding result in the case where $y \geq 0$ and $x > y$. Hence show that if $\mu < 0$, for $y \geq 0$,

$$\lim_{t \to \infty} \mathbb{P}(M_t^\mu \geq y) = e^{2\mu y}.$$

5. (a) Let $X_t = B_t - tB_1$ for $0 \leq t \leq 1$ be the Brownian bridge from 0 to 0 in time 1. Explain why the process $X = (X_t)_{t \in [0,1]}$ is a Gaussian process and compute its mean and covariance function $\mathrm{Cov}(X_t, X_s)$ for $0 \leq s, t \leq 1$.

(b) Show that the process Y defined by setting $Y_t = (1-t)B_{t/(1-t)}$ for $0 \leq t \leq 1$ has the same law as X.

(c) Let $Z = (Z_t)_{t \in [0,1]}$ be the solution to the stochastic differential equation

$$Z_t = Z_0 - \int_0^t \frac{Z_s}{1-s} ds + B_t.$$

By considering the process $U_t = Z_t/(1-t)$ write Z_t as a stochastic integral of a suitable function against Brownian motion.

(d) By using a version of Lévy's characterisation of Brownian motion, or otherwise, show that, if $Z_0 = 0$, then Z has the same law as X.

(e) Show, using the previous question, that

$$P(\sup_{0 \leq t \leq 1} Z_t > x) = \exp(-2x^2).$$

6. Let $B = (B_t^1, \cdots, B_t^n)_{t \geq 0}$ be a standard BM in $\mathbb{R}^n$. Let

$$X_t = \sqrt{(B_t^1)^2 + \cdots + (B_t^n)^2}.$$

Find the SDE for X. By using Levy's characterization show that

$$X_t = X_0 + \int_0^t \frac{n-1}{2X_s} ds + W_t,$$

where W is a Brownian motion in $\mathbb{R}$ (You may assume Ito can be applied).

Let $n = 3$ and let B_0 be a random variable in $\mathbb{R}^3 \backslash \{0\}$, independent of $(B_t - B_0)_{t>0}$.

(a) Show that $1/||B_t||$ is a local martingale, where

$$||(x_1, x_2, x_3)|| = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

(b) Suppose $B_0 = y$. Let $M_t = ||B_{t+1} - y||^{-1}$, for $t \geq 0$. Show by a direct calculation that $E(M_t^2) = \frac{1}{t+1}$. Deduce that M is bounded in L^2 and uniformly integrable. *You may assume that* $P[\forall t > 0, B_{t+1} = y] = 0$.

(c) Show that M is both a local martingale and a supermartingale.

(d) Use the martingale convergence theorem to show that M is *not* a martingale.

7. Let $(W_t)_{t \geq 0}$ be a standard Brownian motion in $\mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, (\mathcal{F})_t, \mathbb{P})$. Consider the stochastic differential equation (SDE) given by

$$dY_t = 3Y_t^2 dt - 2Y_t^{3/2} dW_t, \quad Y_0 = 1. \tag{1}$$

(a) Show that this SDE satisfies a local Lipschitz condition but does not satisfy the linear growth condition.

(b) If $\tau = \inf\{t > 0 : Y_t = \infty\}$ denotes the explosion time of Y, find $P(\tau > t)$ and hence show that $\tau < \infty$ almost surely but $\mathbb{E}\tau = \infty$. [*You may find it helpful to apply Ito's formula to processes of the form* $(a + W_t)^\alpha$.]

8. Let $\sigma_\alpha(x) = |x|^\alpha \wedge 1$ for $x \in \mathbb{R}^2$ and let $X \in \mathbb{R}^2$ satisfy

$$dX_t^i = \sigma_\alpha(X_t) dW_t^i, \quad X_0^i = 0, \quad i = 1, 2$$

where W is a Brownian motion in $\mathbb{R}^2$.

(a) Find a (trivial) solution to the SDE for all $\alpha > 0$. Show that if $\alpha \geq 1$, this is the unique strong solution.

(b) If $\alpha < 1$, we show that we can time change Brownian motion to give another solution:

i. Show that $\langle X^i \rangle$ does not depend on i and hence that if $\tau_t = \inf\{u : \langle X^i \rangle_u > t\}$, then we can write

$$\tau_t = \int_0^t \sigma_\alpha^{-2}(W_s) ds.$$

ii. By computing $\mathbb{E}(\tau_t)$, show that this time change is finite almost surely for $\alpha < 1$ and hence we can obtain another solution X via the Dubins-Schwarz Theorem.