Foundations of Stochastic Analysis

The basic building block is <u>Brownian motion</u>

Observation by Brown 1823

First theory Einstein 1905

Wiener proved Einstein's theory 1920s

Levy proved many properties 1930, 40s

<u>Definition</u>

A stochastic process $W = (W_t)_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard Brownian motion in $\mathbb{R}$ if

(1) $W_t - W_s \sim N(0, t-s) \quad \forall\, 0 \leq s < t$

(2) $W_t - W_s$ is independent of $W_s \quad \forall\, 0 \leq s < t$

(3) $W_0 = 0$, $t \to W_t$ is continuous

A B.M. in $\mathbb{R}^d$ is just d independent 1-d B.M.s

An alternative definition

Defn

W is a B.M. on $(\Omega, \mathcal{F}, \mathbb{P})$

if it is a cts <u>Gaussian process</u>

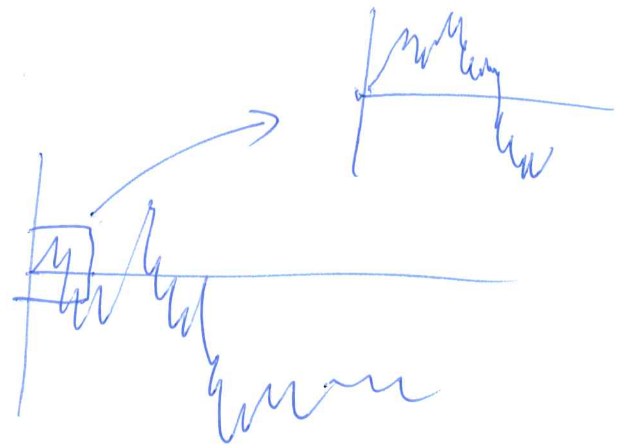
with mean 0 and covariance $\min(t,s)$

To show this is equivalent is an easy calculation

Properties

(1) Scaling: Fix $\lambda > 0$

then $B_t = \lambda W_{t/\lambda^2}$ is a B.M.



(2) Time inversion:

$B_t = t W_{1/t}$ is a B.M.

(3) Time reversal:

$B_t = W_1 - W_{1-t}$ is a B.M. $0 < t < 1$

To see these properties use the 2nd definition

– calculate the covariance function

<u>Existence:</u> See the notes for one construction

Wiener used Fourier series

Lévy has a construction via dyadic approximation

Path Properties

B.M. paths are continuous but highly irregular

(1) $\mathbb{P}\left(\sup_{t>0} W_t = \infty \right) = \mathbb{P}\left(\inf_{t>0} W_t = -\infty \right) = 1$

(2) To be differentiable at 0 we would need

$$\lim_{t \downarrow 0} \frac{W_t}{t} \quad \text{exists}$$

<u>But</u> by time inversion $\lim_{t \downarrow 0} \frac{W_t}{t} = \lim_{s \to \infty} s W_{1/s}$ $\qquad t = 1/s$

$$= \lim_{s \to \infty} B_s$$

So by (1) this limit doesn't exist

$$\mathbb{P}\left(\limsup \frac{W_t}{t} = \infty\right) = \mathbb{P}\left(\liminf \frac{W_t}{t} = -\infty\right) = 1$$

More is true : $\mathbb{P}$(W is not differentiable at any pt in $[0,1]$) $= 1$

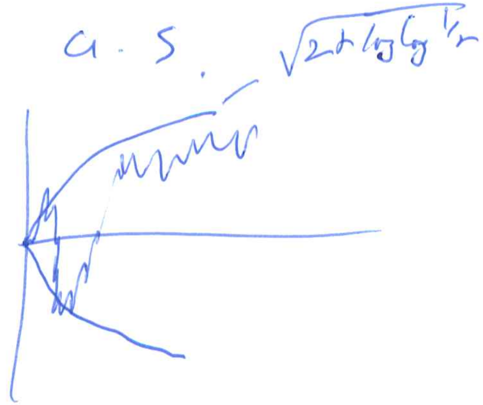
(3) Precise information about the path regularity :

$$\limsup_{\delta \downarrow 0} \sup_{0 \le t \le 1} \frac{|W_{t+\delta} - W_t|}{\sqrt{2\delta \log 1/\delta}} = 1 \quad \text{a.s.}$$

W is Hölder continuous of order $1/2 - \varepsilon$

(4) LIL — Law of Iterated Logarithm

$$\limsup_{t \downarrow 0} \frac{W_t}{\sqrt{2t \log\log 1/t}} = 1 \quad \text{a.s.}$$

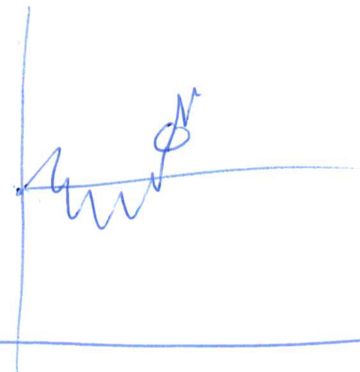


time inversion :

$$\limsup_{t \downarrow 0} \frac{t\, W_{1/t}}{\sqrt{2t \log\log 1/t}} = \limsup_{t \to \infty} \frac{W_t}{\sqrt{2t \log\log(t)}} = 1$$

(5) There are no points of increase

$$\mathbb{P}(\exists \delta > 0, s > 0 \text{ s.t. } W_{s-h} \le W_s \le W_{s+h} \ \forall h \in [0,\delta]) = 0$$



Definition: A stopping time T is a random time s.t.

$$\{T \le t\} \in \mathcal{F}_t = \sigma(W_s : s \le t)$$

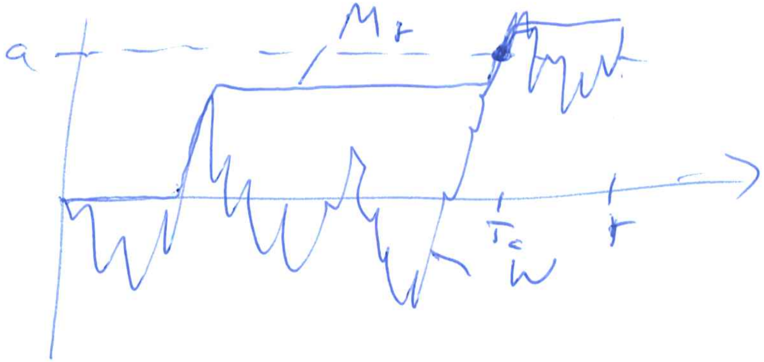
$\{\mathcal{F}_t\}_{t \ge 0}$ is called the natural filtration

E.g. $T_a = \inf\{t \ge 0 : W_t = a\}$

hitting time

Consider the maximum process

$$M_t := \sup_{0 \le s \le t} W_s$$



$$\{M_t > a\} = \{T_a < t\} \qquad (*)$$

$$\mathbb{P}(W_t \geq a) = \mathbb{P}(W_t \geq a, T_a \le t)$$

$$= \mathbb{P}(W_t \geq a \mid T_a \le t)\, \mathbb{P}(T_a \le t)$$

$$= \mathbb{P}(W_{t-T_a} > 0 \mid \cancel{T_a \le t})\, \mathbb{P}(T_a \le t) \quad \text{by } (*)$$

Strong Markov property - Markov property holds at stopping time

$$= \frac{1}{2}\mathbb{P}(M_t \geq a)$$

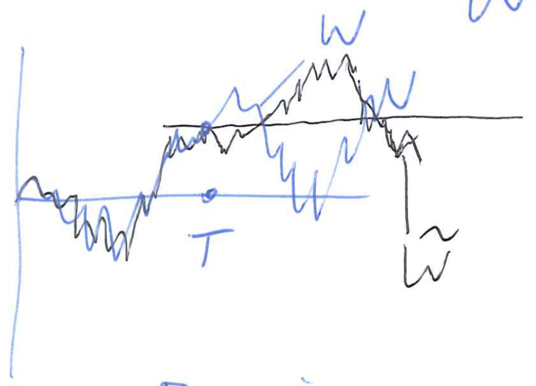
$$\therefore \mathbb{P}(M_t \geq a) = 2\,\mathbb{P}(W_t \geq a)$$

The reflection principle

By symmetry of Brownian motion

If T is a stopping time

$$\tilde{W}_t = \begin{cases} W_t & t < T \\ 2W_T - W_t & t \geq T \end{cases} \quad \text{is a B.M.}$$



Exercise: use this to show $(T = T_b)$

$$\mathbb{P}(M_t \geq b, W_t \leq a) = \mathbb{P}(W_t \geq 2b - a)$$

Filtration

Definition

A filtration is an increasing sequence of σ-algebras $(\mathcal{F}_n)_{n\geq 0}$ $\quad \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$

We can define $\mathcal{F}_T = \{A : A \cap \{T=n\} \in \mathcal{F}_n\}$

information available up to the stopping time T

Uniform Integrability

$\mathbb{E}|X| < \infty$

Defn A family of integrable random variables $\mathcal{A}$ is UI if

$$\lim_{N\to\infty} \sup_{X\in\mathcal{A}} \int_{|X|>N} |X| \, d\mathbb{P} = 0$$

i.e. for $\varepsilon > 0$ $\exists N$ s.t. $\mathbb{E}|X| \mathbb{1}_{|X|>N} < \varepsilon \quad \forall X \in \mathcal{A}$

Condition for UI

(1) If $\sup_{X \in \mathcal{A}} \mathbb{E}|X|^p < \infty$ for some $p > 1$ $\Rightarrow$ $\mathcal{A}$ is UI

(2) If $\exists\, Y$ s.t. $|X| \leq Y$ $\forall X \in \mathcal{A}$ and $\mathbb{E}|Y| < \infty$
$\Rightarrow$ $\mathcal{A}$ is UI

Theorem

If $X_n \to X$ in prob and $\{X_n\}$ is UI then

$X_n \to X$ in L^1

e.g. Let $X_n = \begin{cases} 0 & \text{prob } 1 - 1/n \\ n^\alpha & \text{prob } 1/n \end{cases}$

check X_n is not UI if $\alpha \geq 1$

Martingales

Discrete time

Definition

$\{ M_n : n \geq 0 \}$ is a martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$

if (1) M_n is $\mathcal{F}_n$ - measurable (M_n is $\mathcal{F}_n$ adapted)

(2) M_n is integrable $\mathbb{E}|M_n| < \infty \quad \forall n$

(3) $\mathbb{E}(M_{n+1} | \mathcal{F}_n) = M_n \quad \forall n$

Clearly taking $\mathbb{E}$ in (3) $\implies \mathbb{E} M_{n+1} = \mathbb{E} M_n = \cdots = \mathbb{E} M_0$

Classical example: symmetric random walk

$$M_n = \sum_{i=1}^{n} X_i \qquad X_i \sim \text{iid } \mathbb{E}|X_i| < \infty \text{ mean } 0$$

M is a sub martingale

(3) $\implies \mathbb{E}(M_{n+1} | \mathcal{F}_n) \geq M_n$

super martingale

(3) $\longrightarrow$ $\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) \le M_n$

~~M~~ Key Theorem

1. Optional Stopping Theorem:

(1) If T is a bounded stopping time and M is a martingale

then $\mathbb{E} M_T = \mathbb{E} M_0$

(2) If M is a UI martingale and S, T stopping time

$$\mathbb{E}(M_T \mid \mathcal{F}_S) = M_S$$

2. Martingale Convergence Theorem

If M is L^1-bdd $\sup_n \mathbb{E}|M_n| < \infty$

then $M_n \longrightarrow M_\infty$ a.s. as $n \to \infty$

If M is UI then $M_n \longrightarrow M_\infty$ in L^1 and a.s.

and $\mathbb{E}(M_\infty | \mathcal{F}_n) = M_n$

3. <u>Doob's inequalities</u>

If M is a super martingale

$$\mathbb{P}\left(\sup_{0 \le k \le n} M_k > \lambda \right) \le \frac{\mathbb{E} M_0}{\lambda} \qquad \forall \lambda > 0$$

If M is a martingale for any $p > 1$

$$\mathbb{E}\left[\sup_{0 \le k \le n} | M_k |^p \right] \le \left(\frac{p}{p-1} \right)^p \mathbb{E} |M_n|^p$$

Doob's L^p-inequality

e.g. $p = 2$ $\quad \mathbb{E} \sup_{0 \le k \le n} |M_k|^2 \le 4 \mathbb{E} M_n^2$

Proofs of these are sharper versions in the notes

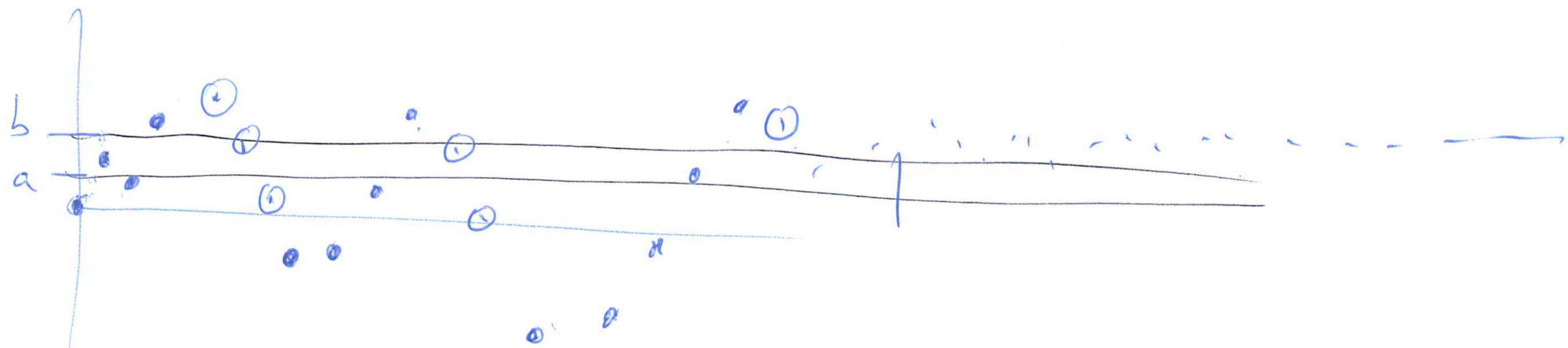
Proof of MCT

We use the martingale transform:

If $(C_n)_{n\geq 1}$ is a previsible process $C_n \in \mathcal{F}_{n-1}$, M_n is a (super)martingale

then $X_n = (C \cdot M)_n = \sum_{i=1}^{n} C_i (M_i - M_{i-1})$

is a (super) martingale



Let $U_a^b(M; n)$ = # of ~~upcros~~ upcrossings from a to b by M in time n

let $C_1 = 1(X_0 < a)$ $C_n = 1_{(C_{n-1}=1)} 1_{X_{n-1} \leq b} + 1_{(C_{n-1}=0)} 1_{(X_n < a)}$

$$X_n = (C \cdot M)_n \geq U_a^b(M,n)(b-a) - (X_n - a)^-$$

$\Rightarrow$ X_n is a martingale

$$\mathbb{E} X_n = 0$$

$$\therefore \quad \mathbb{E}\, U_a^b(M,n)\,(b-a) \leq \mathbb{E}(X_n - a)^-$$

$$\text{i.e.} \quad \mathbb{E}\, U_a^b(M,n) \leq \frac{\mathbb{E}|X_n| + a}{b-a}$$

hence if X_n is L^1-bdd $\Rightarrow$ $\mathbb{E}\, U_a^b(M) < \infty$

$$U_a^b(M) = \lim_{n\to\infty} U_a^b(M,n)$$

this is enough to give MCT

Continuous Martingales

Filtration: Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration: We say it satisfies the usual condition if

(1) $\mathcal{F}_t$ is $\mathbb{P}$-complete – if $B \subset A \in \mathcal{F}_t$ $\mathbb{P}(A)=0$
$\Rightarrow B \in \mathcal{F}_t$ and $\mathbb{P}(B)=0$

(2) $\mathcal{F}_t$ is right continuous

$$\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$$

Definition

X is progressively measurable w.r.t. $(\mathcal{F}_t)$ if

$\forall s$ $(s,\omega) \longrightarrow X_s(\omega)$ is $\mathcal{B}([0,s]) \times \mathcal{F}_s$

Proposition

(1) Adapted processes $X_t \in \mathcal{F}_t$ with ~~cts~~ paths ~~cad~~ are progressively measurable

Martingales

A family of random variables $(M_t)_{t \in \mathbb{R}_+}$ is a cts time martingale if

(1) M_t is $\mathcal{F}_t$ - adapted

(2) $\mathbb{E}|M_t| < \infty \quad \forall t$

(3) $\mathbb{E}(M_t | \mathcal{F}_s) = M_s \quad 0 < s < t$

Examples: W_t, $W_t^2 - t$, $\exp(W_t - \frac{1}{2}t)$

The main results results in continuous time carry over from discrete time. One new issue is path regularity