ANALYSIS I

A Number Called e

These supplementary notes by H A Priestley provide a (non-examinable) proof of the useful fact that

$$\mathrm{e} = \left(1 + \frac{1}{n}\right)^n.$$

[An alternative, and simpler, proof of the more general result in which $x \in \mathbb{R}^{>0}$ replaces n can be based on L'Hôpital's Rule (in Analysis II).]

e.1. The number e is defined to be

$$\mathrm{e} = \sum_{k=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots ,$$

where, by convention, $0! = 1$. Problem sheet 5, Q. 5, asked for a proof that the partial sum sequence of the series above is monotonic increasing and bounded above. Hence it converges to a real number, so that e is well defined. You were also asked to show e is irrational.

Problem sheet 1, Q.5, introduced sequences

$$\alpha_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad \beta_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

and asked for a proof that, for all n,

$$\alpha_n \leqslant \alpha_{n+1} \leqslant \cdots \leqslant \beta_{n+1} \leqslant \beta_n.$$

Example 6.3(c) then applied the Monotonic Sequences Theorem to prove that (α_n) converges.

We now provide the desired reconciliation.

e.2 Proposition.

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Proof. Let $s_n = \sum_{k=0}^{n} \frac{1}{n!}$.

By the Binomial Theorem

$$\begin{aligned}\alpha_n &= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots + \frac{1}{n^n}\\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\frac{1}{n}\\ &\leqslant 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = s_n.\end{aligned}$$

From this we have $\lim \alpha_n \leqslant \mathrm{e}$.

On the other hand, if m, n are natural numbers with $m < n$, focusing on the first $m+1$ terms of α_n we see that

$$1 + 1 + \left(1 - \frac{1}{n}\right)\frac{1}{2!} + \cdots + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right)\frac{1}{m!} \leqslant \alpha_n.$$

If we fix m and let $n \to \infty$ then we have, using the Algebra of Limits and recalling that limits respect weak inequalities,

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} \leqslant \lim \alpha_n.$$

Finally letting $m \to \infty$ we have $\mathrm{e} \leqslant \lim \alpha_n$ and the result follows. □

e.3 Another useful limit.

$$\left(1 - \frac{1}{n}\right)^n \to \frac{1}{\mathrm{e}}.$$

Proof. See Problem sheet 5, Q. 6. □