

# Lecture 5, Sci. Comp. for DPhil Students

Nick Trefethen, Tuesday 29.10.19

## Today

### II. Dense linear algebra

- II.1 Matrices, vectors and expansions
- II.2 Orthogonal vectors and matrices

## Handout

- `m07_expansions.m` and `m08_least_squares.m`
- Assignment 2

## Announcements

Assignment 2 due is next Tuesday.

These lectures go by easily and I hope are entertaining enough. Nevertheless we are covering important ground fast. I'd like to emphasize that if you are serious about learning this material, you must take time to study lecture notes, M-files, and reading materials:

<https://courses.maths.ox.ac.uk/node/45032>

Four excellent references on iterative linear systems and eigenvalues, available online: Saad 1 & 2, Templates 1 & 2. See the Books and Journals handout or `tools.html` for links.

## II. Dense linear algebra

We've had four lectures on large-scale sparse linear algebra: matrices of dimensions in the tens of thousands or more.

For the next four lectures, I want to scale back to the fundamental notions and algorithms of dense numerical linear algebra. You may think you know this material already, but I want to emphasise certain points of view that may not be so familiar to you.

The ways of thinking that we will talk about have proved fruitful for linear and nonlinear problems all across science and engineering.

Good reading: Trefethen & Bau chapters 1–3, pp. 3–24

### II.1 Matrices, vectors and expansions

I want you to think about matrix-vector products columnwise:







Entrywise, this is obvious:  $x_j = q_j^T b$ .

**Proposition.** *If  $Q$  is orthogonal, then  $\|Qx\| = \|x\|$ .*

*Proof.*

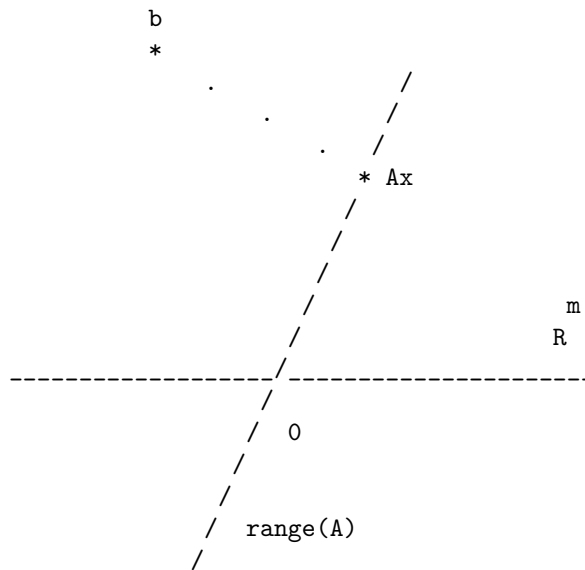
$$\|Qx\|^2 = (Qx)^T(Qx) = x^T Q^T Q x = x^T x = \|x\|^2$$

Suppose  $A$  is  $m \times n$  with  $m > n$ .

A **least-squares solution** to  $Ax = b$  is a vector  $x$  such that

$$\|Ax - b\| = \text{minimum} \quad (*)$$

Let's draw a picture (which is supposed to show a right angle):



It's clear geometrically, and easily proved, that (\*) is equivalent to the condition that the **residual**  $b - Ax$  is orthogonal to  $\text{range}(A)$ .

That is, for each column  $a_j$  of  $A$ ,  $a_j^T(b - Ax) = 0$ . Equivalently,

$$A^T(b - Ax) = 0.$$

These are known as the **normal equations**,

$$A^T b = A^T A x.$$

In Matlab,  $x = A \backslash b$  computes a least-squares solution if  $m > n$ . We'll see how it does it in the next lecture.

[m08\_leastquares.m]

(If extra time, Chebfun quasimatrices)