Lecture 3, Sci. Comp. for DPhil Students II

Nick Trefethen, Tuesday 28.01.20

Last lecture

- IV.5 Order of accuracy
- IV.6 Convergence and stability
- IV.7 Adaptive ODE codes

Today

- IV.8 Planetary motions
- IV.9 Chaos and Lyapunov exponents
- IV.10 The Lorenz equations
- IV.11 Sinai billiards and the SIAM 100-digit challenge

Handouts

- Assignment 1 solutions
- `m29_planets.m` and JPL Solar System Dynamics lab
- First pages of Lorenz's 1963 paper on chaos and SIAM 100-digit challenge
- `m30_circles.m` will be executed but not handed out

Assmt 1 due now. Assmt 2 will be handed out on Thursday.

IV.8 Planetary motions

Determining the motions of planets, moons, and spacecraft gives beautiful examples of ODE calculations.

Assignment 1 concerned a planet orbiting in a complicated force field. This is a straightforward application of any ODE code. Here for fun is is a movie version of another dynamical problem: three planets initially at rest at the corners of a 3-4-5 right triangle. See also the Chebfun Example "Pythagorean planets".

There are also more specialized methods for such problems — 2nd-order formulas (e.g. "Störmer formulas"), higher orders of accuracy, and "symplectic" formulas to conserve certain quantities and reduce the size of the errors. (See 2006 book *Geometric Numerical Integration* by Hairer, Lubich & Wanner.)

[`m29_planets.m`] [also `m29b_planets`, with $3 \to 3.001$, to $t = 200$]

My favorite example of ODE calculations in action:

When the newspaper predicts an eclipse, where does the data come from?

It seems many organisations are involved, such as
Admiralty Office
Royal Greenwich Observatory
British Astronomical Society
NASA Goddard Space Center
US Naval Observatory
Bureau des Longitudes, France
Indian Dept. of Meteorology
Astronomical Division, Dept. of Hydrography, U. of Tokyo

But almost all the data used by these ultimately came from one place: the Solar System Dynamics Group, Jet Propulsion Laboratory, Pasadena, California (`http://ssd.jpl.nasa.gov/`).

The French appear to be an exception – as of the 1990s, at least, they did the calculations themselves. I don't have up-to-date information.

Some of the key people at SSD are Myles Standish, Jon Giorgini, and Don Yeomans. The system tracks 932326 asteroids, 3607 comets, 209 planetary satellites, 8 planets, sun, various spacecraft, and it's available online.

Numerical method: variable order, variable step-size integrator. (2nd order formulation, related to Störmer formulas.)

Check out the web site!

Relativistic effects are included for many of the bodies.

How accurate? Depends on body and time scale. Present locations are known to centimeters for the moon, meters for the inner planets. Results for a few decades away from present: accuracy of earth and moon to meters, other bodies tens of meters.

The models track both forward and backward thousands of years.

IV.9 Chaos and Lyapunov exponents

[See Chapter 13 of *Exploring ODEs*]

Let us ask: if the initial data are perturbed infinitesimally, how fast can the perturbation grow with t?

For some dynamical systems, the answer is: at most algebraically, i.e., at a rate $O(t^n)$ for some n.

A **chaotic** dynamical system is one whose perturbations may grow exponentially, i.e., at a rate $O(\exp(Ct))$ for some $C > 0$. The maximum such C is the **Lyapunov exponent** for the system. (In addition, to be chaotic, the sytem should exhibit irregular, aperiodic long-term behaviour with bounded trajectories.)

A chaotic system of ODEs is deterministic in principle, but unpredictable in practice beyond a certain time.

In phase space (i.e. the space of components $u^{(k)}$ of a 1st-order system) its orbits may be tangled up in a **strange attractor**.

The possibility of exponential growth of perturbations seems obvious to anyone who plays with codes like `m29_planets`. But playing around was not so easy in the past. Chaos was lurking in the work of Poincaré around 1900. However, it did not become widely appreciated until work in the 1970s of Ruelle & Takens, Feigenbaum, May, Yorke, et al., following an epochal paper of Ed Lorenz in 1963, one of the great scientific papers of the 20th century.

Bob May eventually became Lord May and President of the Royal Society in good part because of this work. Lorenz died in 2008 at age 90. Today Google gives 445m hits on "chaos".

[First page of Lorenz's paper. 21694 citations at Google Scholar.]

Examples

Billiards on square table: not chaotic.

Billiards on a curved table: for most tables, chaotic.

Any linear equation: not chaotic.

Any 1D or 2D 1st-order ODE, like the van der Pol eq: not chaotic, because the phase space is a plane, and orbits cannot get tangled.

Weather on planet earth (Lorenz's motivation) essentially chaotic, though not a precisely defined mathematical system.

And here's a heavily studied example: turbulent flow in a pipe. This dynamical system appears chaotic at higher Reynolds numbers R, though one theory holds that in principle it's just "transient chaos" with a spectacularly long time scale (something like $\exp(\exp(R))$) (Hof et al. 2008). For a garden hose ($R = 2400$) relaminarization would take 5 years and 40000 km. That's a long pipe!

There are lots of books about chaos and nonlinear dynamics. Everyone's favorite: Strogatz. [Pass it around] Strogatz has some videos on chaos that are very popular; you can see a sample at his web site.

We are speaking casually, but these ideas have all been made rigorous.

Now, what about our 3-planets problem? Further simulations show that it's chaotic, with Lyapunov exponent something like 0.14.

There is computational evidence that the solar system is chaotic too (time scale tens of millions of years), though there is some controversy. Some key names: Sussman & Wisdom, 1990s; Laskar & Gastineau more recently.

Control of chaos. If perturbations have big effects, then big effects may be achieved by small perturbations. For example, consider our 3-planets problem. Could you perturb the initial positions by an amount too small to see by eye and have the effect that all three planets exactly collide at a time approximately 50? I suspect so. (3-planet collision: four equations. Initial position of the planets: three parameters, after you dispose of translation and rotation. A fourth parameter would come from permitting collision at time "approximately" 50 rather than "exactly".)

I would be delighted if somebody figures this out and makes a demo for us all to enjoy.

IV.10 The Lorenz equations

Here's another example - the **Lorenz equations** from 1963 - a classic chaotic system of 3 ODEs, originally motivated by convection in the atmosphere. These equations are famous, and there's at least one book about them (Sparrow, 1982).

With standard parameters:

$$x' = 10y - 10x, \qquad y' = 28x - y - xz, \qquad z' = xy - (8/3)z$$

How do these equations behave? Well, $x = y = z = 0$ is one solution (a saddle point), not exciting.

There are also a pair of unstable fixed points:

$$x = y = \pm 6\sqrt{2}, \quad z = 27.$$

Experiments show seemingly random behaviour of $x(t)$ or $y(t)$ (sketch). Imagine how hard it was for Lorenz in 1963 to persuade people that this was not caused by errors in his computation!

In phase space, the solutions lie on a strange attractor of dimension about 2.0627160. The strange attractor was conjectured by Lorenz and finally proved with computer assistance by Warwick Tucker ("The Lorenz attractor exists", *Comptes Rendus*, 1999). The dimension estimate comes from Divakar Viswanath (Physica D 2004).

The Lyapunov exponent for this system is about 0.9057, i.e., trajectories may diverge at an average rate around $\exp(0.9t)$.

Different constants than 10, 28, 8/3 give different Lyapunov exponents.

Demonstration in MATLAB: `lorenz.m` [a built-in demo]

Demonstration in Chebfun: Chebgui. Try changing 28 to 22.

For further equations with chaotic behavior see Examples 64-70 of Appendix B of *Exploring ODEs*.

IV.11 Sinai billiards and the SIAM 100-digit challenge

Here's another example of a chaotic problem – problem 2 of the "Hundred-dollar, hundred-digit challenge" (*SIAM News*, Jan/Feb 2002 and Jul/Aug 2002).

`[100-dollar challenge handout]`

Suppose we have an infinite lattice of circles in the x-y plane with a ball bouncing off them [sketch]. This is equivalent to the so-called **Sinai billiard**.

Demonstration in MATLAB

[`m30_circles.m`]

Because of the chaos, one can't compute ten digits of accuracy in ordinary 16-digit precision.

Such problems raise interesting questions about numerical computation. On the one hand, it's clear that you need extra precision to get a numerically correct answer. On the other hand, it's not clear that there could ever be a true scientific reason for wanting that numerically correct answer. "If the answer is highly sensitive to perturbations, you have probably asked the wrong question."

In the 100-digit challenge, this was the only one of the ten problems that was chaotic; it was also the only one to require extended precision.