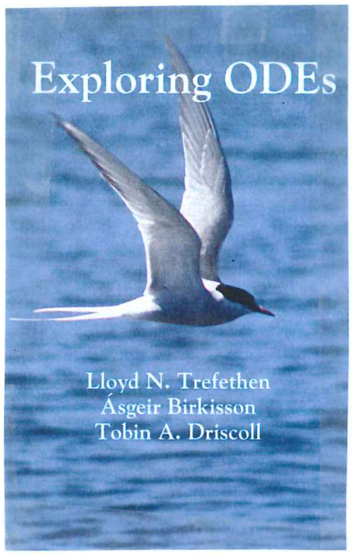
Scientific Computing for DPhil Students II

Assignment 2

Due at lecture at 10:00 on Tuesday, 10 February 2020.

These problems are from *Exploring ODEs,* which is freely available online from `https://people.maths.ox.ac.uk/trefethen/`. All but the final problem can be solved in a few lines of Chebfun; be sure to include listings of your code. If you prefer to use Matlab, that is fine too.

Exercise 5.9. Water droplet. The height of the surface of a water droplet satisfies $y'' = (y-1)(1+(y')^2)^{1.5}$ with $y(\pm 1) = 0$. What is the height at the midpoint $x = 0$?



Exercise 8.1. Exploiting resonance to increase amplitude. For a given frequency ν, consider the solution to $y'' + y = 1 - \cos(\nu t)$, $y(0) = 1$, $y'(0) = 0$. *(a)* If ν is set equal to the resonant frequency ω for this equation, compute the time t_c at which $y(t)$ first reaches the value 10. *(b)* What is the smallest value of ν for which $y(t)$ reaches the value 10 at some time $t \in [0, 100]$?

Exercise 13.3. Alternative choices of the Lorenz coefficient 28. In the Lorenz equations, let r denote the parameter that traditionally takes the value 28. Starting from our usual initial conditions, make plots of $u(t)$ against $w(t)$ as in Figure 13.1 for $t \in [0, 100]$ with $r = 20, 22, 24$; also make plots in each case of $u(t)$ against t as in Figure 13.2. Which case seems to be chaotic? Which one gives the clearest example of transient chaos?

Exercise 15.5. A cyclic system of three ODEs (adapted from Guckenheimer and Holmes, "Structurally stable heteroclinic cycles," *Mathematical Proceedings of the Cambridge Philosophical Society,* 1988). Consider the system of ODEs $u' = u(1 - u^2 - bv^2 - cw^2)$, $v' = v(1 - v^2 - bw^2 - cu^2)$, $w' = w(1 - w^2 - bu^2 - cv^2)$, where b and c are parameters. *(a)* Plot the solution $u(t)$ for $t \in [0, 800]$ with $b = 0.55$, $c = 1.5$ and initial conditions $u(0) = 0.5$, $v(0) = w(0) = 0.49$. Make similar plots of $v(t)$ and $w(t)$, and also of the whole trajectory in u-v-w space, and comment on these shapes. *(b)* What are the four fixed points of this system that the plots just drawn come close to? For large t, the trajectory moves approximately in a cycle from one fixed point, to another, to a third, and then back again. (It is approximating a *heteroclinic cycle.*) Which fixed point is the trajectory near at $t = 800$? *(c)* Find the eigenvalues of the appropriate matrix at one of these fixed points. What does this tell us about the structure of this fixed point? How does that fit with the observed trajectory?

Exercise 16.2. Bounce pass. A ball is thrown from player A to player B, 5 meters away, starting and finishing at height 1 meter. This is an idealized ball that travels as a point mass with no air resistance or rotation and bounces perfectly with equal angles and speeds of impact and rebound. The pass is a slow one: it takes a full 3 seconds to get from A to B. *(a)* Assuming the ball does not bounce, sketch its trajectory. You do not need to write any differential equations. *(b)* Assuming the ball bounces once, sketch all of its possible trajectories. Again you do not need to write any differential equations. *(c)* Now consider all possible solutions to this BVP, with any number of bounces. Assume it takes 0.45 seconds for a point mass to fall from a height of 1 meter. Exactly how many solutions are there all together? Sketch them.