Modelling and Analysis of Continuous Real-world Problems

Sheet 1 - MT 2019

1. Consider a simple model of the heating of a sawdust pile or a compost. The temperature changes in the pile due to heating as the material oxidises. Assume that there is plenty of oxygen and plenty of fuel so that the reaction is only a function of the temperature. Assume that the pile is one dimensional.

   Discuss why a suitable nondimensional model is

   $$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \lambda e^T \ .$$

   with

   $$T(x,0) = 0 \qquad T(0,t) = 0 \qquad T(1,t) = 0 \tag{1}$$

   and briefly explain what scalings are necessary to get this.

   Consider the behaviour of the problem (1) for small time (take $t = \epsilon \hat{t}$ and $T = \epsilon \hat{T}$ as $\epsilon \to 0$) and find the first two terms of the expansion of T in ϵ. Is this approximation valid as $t \to \infty$?

   Consider a simplified version of the problem (1)

   $$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \lambda(1 + T) \ .$$

   with

   $$T(x,0) = 0 \qquad T(0,t) = 0 \qquad T(1,t) = 0$$

   and find the solution. What is the long time behaviour? Are there any critical values of λ?

   Consider the problem (1) in the following cases:
   i) with no spatial variation (neglect the boundary conditions), and
   ii) with no time variation (neglect the initial data).

   From the behaviour in the two cases can you identify what behaviour you might expect from the full problem (1)? Do you expect there critical values of λ for this problem?

2. Use separation of variables to find the solution to

   $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \text{for } t \geq 0, \quad 0 \leq x \leq 1$$

with an initial condition $u(x,0) = x(1-x)$ and boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(1,t) = 0\ .$$

From this solution determine the behaviour as $t \to \infty$.

Without considering the separation of variables solution use the PDE and associated conditions to show directly that

$$\int_0^1 u(x,t)\, dx = \frac{1}{6} \qquad \text{for all } t \geq 0\ .$$

3. Starting with the diffusion equation in three-dimensions and Cartesian coordinates transform the equation into cylindrical polar coordinates (r, θ, z).

4. Consider the variations in temperature of a 1-D rod $0 \leq x \leq a$, with the end $x =$ held at $T = 0$ and the end $x = a$ held at $T = g(t)$, and initial temperature distribution $T = 0$. Derive the solution for the temperature in the form of an infinte sum using Laplace transforms or other methods.

5. Determine if solutions to the following PDE problems are continuously dependent on their initial conditions:

$$(i)\ \ \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 \qquad (ii)\ \ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (iii)\ \ \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2}$$

$$(iv)\ \ \frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad (v)\ \ \frac{\partial u}{\partial t} = \frac{\partial^4 u}{\partial x^4} \qquad (vi)\ \ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$(vii)\ \ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \qquad (viii)\ \ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

6. Consider the one-dimensional diffusion problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \text{for } t \geq 0, \quad 0 \leq x \leq 1$$

with an initial condition $u(x,0) = u_0(x)$ and boundary conditions

$$u(0,t) + \beta_0 \frac{\partial u}{\partial x}(0,t) = g(t), \qquad u(1,t) + \beta_1 \frac{\partial u}{\partial x}(1,t) = h(t).$$

Using the energy method determine what range of values of β_0 and β_1 will give a wellposed problem.

7. Find a similarity solution to the PDE representing heat flow in a relatively cool material

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} .$$

for $0 < x < +\infty$ with

$$T = 0 \text{ at } t = 0, \qquad \frac{\partial T}{\partial x} = -1 \text{ at } x = 0, \qquad T \to 0 \text{ as } x \to +\infty$$

Find a similarity solution to the PDE representing heat flow in a very hot glowing material

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(T^3 \frac{\partial T}{\partial x}\right) .$$

for $0 < x < +\infty$ with

$$T = 0 \text{ at } t = 0, \qquad T^3 \frac{\partial T}{\partial x} = -1 \text{ at } x = 0. \qquad T \to 0 \text{ as } x \to +\infty$$

(you may need to consider doing some numerics to find the solution).

8. Find a similarity solution to the PDE

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^n \frac{\partial h}{\partial x}\right) .$$

for $-\infty < x < +\infty$ where

$$\int_{-\infty}^{+\infty} h(x,t)\, dx = 1$$

and n is a positive constant.

Find the solution when $n = 0$

Discuss how the $n = 0$ solution is different from the case $n > 0$

Briefly consider what happens to the solution if $n < 0$.

9. Examine the stability properties of the problem

$$\frac{\partial c_1}{\partial t} = 14 \frac{\partial^2 c_1}{\partial x^2} - 2c_1 - 2c_2$$

$$\frac{\partial c_2}{\partial t} = 1 \frac{\partial^2 c_2}{\partial x^2} + 2c_1 + c_2$$

with $c_1 = c_2 = 0$ at $x = 0$ and at $x = 2\pi$.

Show that the underlying reaction system, without diffusion, is stable. Determine if diffusion destabilises the system.