Modelling and Analysis of Continuous Real-world Problems

Sheet 2 - MT 2019

1. Assuming that a fluid is inviscid and irrotational and starting with the equations of conservation of mass and conservation of momentum equations (the "Euler" equations) derive the Bernoulli's equation and hence the boundary condition on the free surface of a pool of water.

2. Consider transverse waves on a thin elastic beam (a beam that elastically resists bending) which is also held under tension. The governing equation is

$$\frac{\partial^2 u}{\partial t^2} = \alpha \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial^4 u}{\partial x^4}$$

   Determine both the phase and group velocity of the waves.

3. Show that if a plane wave travelling in an elastic solid only has displacement in the direction of the travel then it must be a P-wave (longitudinal wave). Also show that if a plane wave travelling in an elastic solid only has displacement orthogonal to the direction of the travel then it must be a S-wave (shear wave)

4. Consider Stokes waves in an infinite sea of uniform depth H. Using the dispersion relation show that if the waves are much longer than the depth of water then the waves are not dispersive (linear shallow water waves) while if the waves are much shorter than the depth of water (linear deep water waves) the waves disperse with

$$v_{phase} = 2v_{group}.$$

5. Derive the equation governing transverse waves in one spatial dimension on a string is travelling at constant speed U in the positive x direction.

   Find the general solution to the one space dimension version of the equation for by finding the characteristic variables

$$\xi = x + Bt, \qquad \eta = x + Ct$$

   (where B and C are constants to be chosen) that reduce the problem to canonical form ($u_{\xi\eta} = 0$).

6. There is enormous interest in determining how big a connected forest area needs to be in order to sustain viable populations of certain species. Consider the distribution of squirrels in a large square patch of forest of length W on each side. There is a lake on one side of the forest and the other three sides are farming fields. Squirrels move randomly in the forest with a diffusion coefficient D, they cannot move onto the lake and if they get to a farming field they are either shot or easily eaten by foxes. The squirrels reproduce in the forest at a rate k times their population. Write down a model for $n(x, y, t)$ the population (number of squirrels per m^2) in the forest. Consider the steady state for the squirrels and use separation of variables to find their possible distribution in the forest. Is there always a steady state and are there more than one? What is the minimum size of the forest to sustain the squirrels?

   Discuss how this problem relates to finding the vibration modes of a square membrane that is held firmly on three sides and is free to flap on the fourth side. Find the modes of the vibrations and order the first few by the magnitude of their frequency.

7. Use the Kirchhoff-Helmholtz representation and the method of images to show that the solution of

$$(\nabla^2 + k^2)\phi = 0, \quad \forall x_1, x_2 \text{ and } \quad x_3 > 0$$
$$\phi(x_1, x_2, 0) = u(x_1, x_2)$$

   with out-ward going wave behaviour at infinity is

$$\phi(\mathbf{x}) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(y_1, y_2)\, \frac{\partial G}{\partial y_3}(\mathbf{x}, y_1, y_2, 0)\, dy_1\, dy_2$$

   where

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi}\left(\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} - \frac{e^{ik|\mathbf{x}'-\mathbf{y}|}}{|\mathbf{x}'-\mathbf{y}|}\right)$$

   and where the image point $\mathbf{x}'$ and $\mathbf{x}$ are related by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{x}' = \begin{pmatrix} x_1 \\ x_2 \\ -x_3 \end{pmatrix}.$$