Modelling, Analysis and Computation of Continuous Real-world Problems

Sheet 3 - MT 2019

1. Use the method of characteristics to find the general solution, in explicit form, of the equation
$$x^3 \frac{\partial u}{\partial x} + y^3 \frac{\partial u}{\partial y} = 0.$$

2. Use the method of characteristics to find the solution, in explicit form, of the equation
$$\tan x \frac{\partial u}{\partial x} + \cot y \frac{\partial u}{\partial y} = 0$$
which satisfies $u(x, 0) = \cos^2 x$.

3. Consider transverse vibrations of a string in one space dimension for for $t \geq 0$ and $0 \leq x \leq 1$. Write the problem as a system of 2 first order equations and hence find the projected characteristics.

4. Determine the projected characteristics of the system
$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$
and hence sketch the two families of projected characteristics.

5. Consider the system
$$(t - x)\, u_t + \frac{3}{2} w_t + u_x = 0$$
$$(t - x)\, w_t + \frac{2}{3} x\, u_t + w_x = 0.$$
Show that the projected characteristics are real in the region $x > 0$.

   Suppose the system is to be solved in the region $t > 0$, $\frac{1}{2} < x < 1$. Using the local behaviour of the projected characteristics discuss how many boundary conditions should be imposed at each point on the boundary.

6. Solve the PDE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

for $t > 0$, subject to $u(x, 0) = f(x)$ in each of the following two cases.

(a)

$$f(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x < 1 \end{cases}$$

(b)

$$f(x) = \begin{cases} 0 & x < 0 \\ -x & 0 \leq x \leq 1 \\ -1 & x < 1 \end{cases}$$

In each case, if necessary, allow the solution to be a single-valued weak solution by introducing a shock. For each case sketch the resulting solution $u(x, t)$ versus x for a few t, and the characteristic projections in the (x, t)-plane.