Modelling and Analysis of Continuous Real-world Problems

Sheet 4 - MT 2019

1. Prove the Beltrami identity (see the notes for details) When $\partial f/\partial x = 0$,

$$f - y' \frac{\partial f}{\partial y'} = C, \tag{0.1}$$

where C is a constant.

2. Derive the 2nd order Euler-Lagrange equation (see the notes for details) that minimise

$$\int_{x_A}^{x_B} f(x, y, y', y'') dx \tag{0.2}$$

and is written as

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0, \tag{0.3}$$

stating what boundary conditions are required

3. An industrial oven has a volume of 1 metre squared, and has a square base. It loses heat from its top at twice the rate it loses heat from its sides, and at four times the rate it loses heat from its bottom. Find the dimensions of the most energy efficient oven.

4. * Dido's problem. From Virgil's Aeneid:

"The Kingdom you see is Carthage, the Tyrians, the town of Agenor; but the country around is Libya, no folk to meet in war. Dido, who left the city of Tyre to escape her brother, rules here–a long and labyrinthine tale of wrong is hers, but I will touch on its salient points in order....Dido, in great disquiet, organised her friends for escape. They met together, all those who harshly hated the tyrant or keenly feared him: they seized some ships which chanced to be ready... They came to this spot, where to-day you can behold the mighty battlements and the rising citadel of New Carthage, and purchased a site, which was named 'Bull's Hide' after the bargain by which they should get as much land as they could enclose with a bull's hide."

Dido cut the bull's hide into strips to make a long line of length L. Your problem is to work out what was the shape of the piece of the largest piece of land that she could buy. Assume that Carthage is situated on a flat section of coast.

(i) Show that the problem consists of minimising

$$\int_a^b y dx \tag{0.4}$$

subject to a constraint that you should state.

(ii) Use the Euler-Lagrange equations to derive that

$$\frac{\lambda y''}{(1+y'^2)^{3/2}} = 1. \tag{0.5}$$

(iii) Transform the equations from being in terms of x and y into terms of the angle from horizontal θ, and the arclength s. *Hint:* $dy/dx = \tan\theta$, $dx = ds\cos\theta$... This should dramatically simplify the equation. What shape should Dido have created? What was the area of her new piece of land?

5. An incompressible Mooney-Rivlin solid (such as an elastomer) in uniaxial tension has a strain energy

$$E_{el} = C_1\left(\lambda^2 + \frac{2}{\lambda}\right) + C_2\left(2\lambda + \frac{1}{\lambda^2}\right) \tag{0.6}$$

where $\lambda = l/l_0$, l is the length of the material, and l_0 is a constant.

(i) What is the length of the material when no force is applied to it?

(ii) Work out the equation that describes the solid behaviour when a mass is hung on its end.

6. A circular piece of thin, elastic sheet is placed onto the top of a bath of water. Surface tension will pull the sheet outwards, stretching it. Show that the change in radius is given by

$$\frac{r - r_0}{r} = \frac{\gamma_{lv}(1-\nu)}{hE} \tag{0.7}$$

Hint: The elastic energy of a circular sheet with a line force F acting on its thin edge is

$$E_{el} = \frac{\pi h E}{1-\nu}\left(r - r_0\right)^2, \tag{0.8}$$

where ν is the Poisson's ratio of the sheet (a measure of its compressibility), E is its Young's modulus (a measure of its stiffness), h is the thickness of the sheet, r_0 is its initial radius, and r is its final radius.

If the sheet is made of rubber with $h = 1\mu$m and $r_0 = 10$cm, how much will the sheet expand?

7. Calculate the shape of a beam hanging between two points, which has a non-negligible density ρ. How many boundary conditions do you need to give an exact solution? What extra boundary conditions might you pick to apply?

8. A thin elastic sheet is placed on the surface of water and compressed from its ends. The sheet wrinkles, with no air pockets forming under the wrinkles.

   (i) Assuming a 2d system, with negligible mass of the sheet, derive the governing equation that describes the shape of the elastic sheet.

   (ii) We wish to estimate the characteristic lengthscale of the wrinkles. Assume that the surface takes the shape of a sine wave $h = A\sin(2\pi x/\lambda)$. Calculate the bending energy per unit length of the sheet. Calculate the gravitational potential energy of the water per unit length of sheet. Compare these two to obtain the characteristic wrinkling lengthscale l_w.

9. Assume that you have a 2d liquid crystal bounded between two plates $y = 0, d$ with the rod directors constrained to be in the x, y directions.

   (i) Write down an expression for $\mathbf{n}$.

   (ii) We apply a magnetic field pointing in the y direction, $\mathbf{B} = (0, B, 0)$. Assume that $\theta = \theta(y)$, and use (this is in the notes)

$$W = \frac{K_1}{2}(\theta_x^2 \sin^2\theta - 2\theta_x\theta_y \sin\theta\cos\theta + \theta_y^2\cos^2\theta) + \frac{K_3}{2}(\theta_x^2\cos^2\theta + 2\theta_x\theta_y\sin\theta\cos\theta + \theta_y^2\sin^2\theta). \tag{0.9}$$

   to write down the potential energy of the system.

   (iii) Use the Beltrami identity to show that

$$\theta_y^2(K_1\cos^2\theta + K_3\sin^2\theta) + \mu_0^{-1}\Delta\chi B^2\sin^2\theta = c, \tag{0.10}$$

   where c is a constant.

10. The Fréedericksz transition in the twist geometry. Assume that all the directors of the rods are horizontal, so that $\mathbf{n} = (\cos\phi, \sin\phi, 0)$, but with $\phi = \phi(z)$. We make a cell with walls at $z = 0, d$ and $\phi(0) = \phi(d) = 0$.

    (i) Show that this setup allows twist, but bend and splay are always zero.

    (ii) If we apply an electric field in the y direction, show that the potential energy density is

$$\frac{K_2}{2}\phi_z^2 - \frac{1}{2}\epsilon_0\Delta\epsilon E^2\sin^2\phi. \tag{0.11}$$

    (iii) Minimise the energy of the system, and and show that

$$\xi_d^2\phi_{\tilde{z}}^2 + \sin^2\phi = \sin^2\phi_m, \tag{0.12}$$

where $\hat{z} = z/d$ and $\phi_m = \phi(d/2)$.

(iv) Follow through the argument from the Fréedericksz transition in lectures to show that there is also a critical size of electric field where the rods will start to twist to line up with the electric field.