CDT Mathematical Modelling Course Tasks

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1 Modelling Questions

1. Starting from first principles, derive a model to describe the evolution in concentration of a substance which is dissolved in water, moves by diffusion, and is consumed by a reaction which eats it up at a rate proportional to its concentration.

   In the case where the aqueous solution is put into a non-reacting container and initially has uniform concentration throughout, write down appropriate boundary and initial conditions, and solve the model to find how the concentration varies with time.

2. Write down a very simple mathematical model for how a lump of sugar dissolves in a cup of tea. Does it take a finite or infinite time to dissolve? Now suppose that you are given an observation of the volume of the sugar lump at some time, under what circumstances can you determine when the lump was put in the cup?

3. One way of disposing of toxic waste is to seal it in a drum and dropping it into the sea. These drums sometimes crack if they hit the bottom of the sea, releasing their deadly contents. Experiments confirm that standard drums (1 m high, 0.25 m radius) of weight 500 kg including their contents are likely to crack if they hit the bottom of the sea at a speed greater than 4 m/s. Assuming that the forces acting on the drum are its weight, buoyancy (this is equal to the weight of displaced water), and drag (assuming this is proportional to the square of the velocity, with drag constant 153 kg/m), formulate and solve a mathematical model for the sinking of the drum. Is it likely that the drum will crack?

4. Make a continuum model, based on conservation of mass, for traffic flowing along a single-carriageway road, assuming that there is a linear relationship between the density and the speed of the traffic.

   Traffic in a 30 mph zone is stopped at a traffic light, which then turns green (and stays green). Estimate how long will it take for the car 100 m away from the traffic light to start moving?

5. **(Harder open-ended question)** A 2 m tall basketball player throws a ball from the free throw line, which is 15 ft from the backboard. The back of the hoop is 6 inches from the backboard and is 10 ft above the ground. The hoop has diameter of 18 inches. A basketball has circumference 29.5 inches and weighs 22 ounces.

   How should the player pick their ball release speed and angle to ensure that they score?

   (Hint - Use your model to work out a relationship between speed and angle so that the centre of the ball passes through the centre of the hoop. Then think about the speed that the ball should be going when it reaches the hoop. If you're still enthusiastic, start thinking about the radius of the ball and the radius of the hoop etc).

2 Dimensions and Units

1. Work out the dimensions of the following quantities in terms of fundamental quantities [you'll want to keep this list for future use]: acceleration; force; pressure; energy; density; viscosity; surface tension; specific heat capacity; thermal conductivity.

2. Make a table of of the following parameter values in MKS and CGS [you'll want to keep this table for future use]: Surface tension of water; density of water; acceleration due to gravity; atmospheric pressure; diffusivity of salt in water; specific heat capacity of steel; thermal conductivity of steel; density of steel.

3. Estimate the Reynolds number for the following situations,
   (i) A pebble is thrown into a pond
   (ii) Rock convecting in the earth's mantle
   (iii) water flowing into sand on a beach.

3 Dimensional analysis[1]

When you have finished each question, work out what Buckingham Pi has to say about the situation.

1. How far does heat significantly penetrate along a metal bar in time τ, if the bar has density ρ, specific heat capacity c and thermal conductivity k?

2. What is the critical dimensionless parameter which will determine whether or not liquid of density ρ and surface tension γ will flow out of a tube of radius d under gravity?

3. When a raindrop falls into a pond, the shorter waves travel out faster, but when a large stone is dropped in, the opposite is the case. Try and explain these observations assuming that the wavespeed c depends only on the wavelength λ, the water density ρ and (for the raindrop) the surface tension γ or (for the stone) gravity.

4. An instantaneous release of energy from a very small volume creates a rapidly expanding high-pressure fireball bounded by a very strong thin spherical shock wave across which the pressure drops abruptly to atmospheric. The pressure inside the fireball is so great that the ambient atmospheric pressure is negligible by comparison and the only property of the *air* that determines the radius of the fireball is it's density. Determine how the radius varies with time.

5. (**Harder question**) A boat carried N similar people, each of whom can put power P into propelling the boat. Assuming that they each require the same volume V of boat to accommodate them, show that the wetted area A of the boat is proportional to $(NV)^{(2/3)}$. Assuming that the flow of water is inviscid, why is the drag force proportional to $\rho U^2 A$, where U is the speed of the boat and ρ is the density of the water? Using the fact that the total power exerted by the rowers must be equal to the rate at which work is done, figure out how the speed scales with the other parameters. If P and V are proportional to body mass, is size an advantage to rowing?

[1]With thanks to David Acheson for his old o11 lecture notes (for Qs 2-3) and Sam Howison's book (for Qs 4-5)

4 Nondimensionalisation

1. A simple pendulum is acting under gravity and the pivot of the pendulum is providing a damping force proportional to the angular speed. The pendulum is at a given initial position and velocity.

   Derive a mathematical model for this situation, using $\mathbf{F} = m\mathbf{a}$ in radial polar coordinates as the starting point and using an appropriate scalar product. What are the choices for how to scale time in this problem and what is the physical interpretation of each of them? Determine the dimensionless parameters?

2. A liquid with given initial temperature is flowing steadily past a cylinder of metal which is kept at a different (lower) temperature[2]. Assuming that you are given the velocity of the liquid, write down a model to determine the temperature of the liquid, nondimensionalise the model and identify the key nondimensional parameter(s). What does Buckingham Pi have to say about this problem?

3. (**Harder question**) A cold spherical homogeneous egg is placed into cold water. Over a given time, the water temperature is increased linearly to a given temperature, after which it remains at this temperature. Write down a mathematical model describing the temperature in the egg, assuming that, at the shell,

   $$-k\frac{\partial T}{\partial r} = h(T - T_w), \tag{1}$$

   where k is the thermal conductivity, T is the egg temperature, T_w is the water temperature, and h is a heat transfer coefficient. Check that you have enough boundary conditions for your model! Nondimensionalise the model and identify the dimensionless parameters. What can you say about cooking the egg simply by thinking about the parameters?

   Suppose you now put your egg straight into boiling water. What difference does this make to your conclusions?

[2]Probably by flowing a second liquid along it.

5 Asymptotics

1. Suppose
$$\frac{d^2y}{dx^2} = \epsilon y \frac{dy}{dx}, \qquad y(-1) = 1, \qquad y(1) = -1, \tag{2}$$
where ϵ is small. Find the first two terms in the asymptotic expansion of the solution. Solve the equation explicitly and then using mathematica (use NDSolve). Plot the solution for several values of ϵ along with the one-term and two-term asymptotic solutions to compare the results. Plot the numerical and two-term asymptotic solutions for $\epsilon = 1$ and 2. How good is the approximation?

2. Find the leading-order inner and outer approximations to the solution of
$$\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} + y^2 = 0, \qquad y(0) = 2, \qquad y(1) = \frac{1}{2}, \tag{3}$$
where ϵ is small. Solve the equation using mathematica (use NDSolve) and plot the solution for several values of ϵ (eg 0.1, 0.01) along with the inner, outer and composite asymptotic solutions and compare the results. How good is the approximation?

3. Write down the 2-D radially symmetric model for heat flow along a (uniformly thick) bar which has insulated sides and which has one end immersed in a bucket of water and the other end held at constant temperature in a flame and initially the whole bar is at the temperature of the water. Now assume that the bar is long and thin. Nondimensionalise the model, thinking carefully about the appropriate timescale. Find the leading-order solution to the model in the long-thin limit. Write down the equations and boundary conditions at next order, and use these to close the leading-order model. What can you conclude about the leading-order model?

   Suppose now that the thickness of the bar varies along it's length. What form does the insulated sides boundary condition take in this case? Follow the same receipe as above and find the equation satisfied by the leading-order solution.