INTRODUCTION TO QFT 2019: PROBLEM SET 1

JOHN WHEATER

Some of these problems are mine, some inherited from the previous version of this course given by Professor Fendley. They are all instructive and you need to do them to be able to keep up with the course.

Please hand in your answers at the Reception Desk of the Physics Department by 11am on Monday Week 2 (21 October) preferably using the cover sheet on the website.

1. *Practice on canonical transformations.* Suppose that there are two sets of coordinates (p, q) and (p', q') for which the Hamiltonian functions are $H(p, q)$ and $H'(p', q')$ respectively. Letting

$$p = p' + P(p', q'), \qquad q = q' + Q(p', q')$$

a) Show that the Hamiltonian equations of motion take the same form in primed and unprimed coordinates provided

$$\frac{\partial Q}{\partial q'} + \frac{\partial P}{\partial p'} + \frac{\partial Q}{\partial q'}\frac{\partial P}{\partial p'} - \frac{\partial P}{\partial q'}\frac{\partial Q}{\partial p'} = 0$$

(these are called canonical transformations).

b) Deduce that the phase space element $dp\, dq$ is invariant under canonical transformations.

c) Show that infinitessimal canonical transformations are given by

$$Q = -\epsilon \frac{\partial G(p', q')}{\partial p'}, \qquad P = \epsilon \frac{\partial G(p', q')}{\partial q'},$$

where ϵ is a small parameter. (G is called the generator of the transformation.)

2. *Dirac delta function* The Dirac delta function $\delta(x)$ is an infinite spike of weight 1 located at $x = 0$. It is really a distribution, not a regular function, and strictly only makes sense inside integrals – which means that the relationships discussed below are only supposed to be true inside integrals. For a function $f(x)$ that is sufficiently well-behaved in the region of $x = 0$,

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0).$$

By considering integrals of this form show that

a) $\delta(ax) = a^{-1}\delta(x)$;

b) $\delta(g(x)) = |g'(x)|^{-1}\delta(x - x_0)$, where $g(x)$ has a single zero at $x = x_0$.

c) $\delta(x) = \lim_{\epsilon->0} K \frac{\epsilon}{x^2+\epsilon^2}$, and find the value of K (in this and the next one, put the form of the function inside the integral and the $\lim_{\epsilon\to 0}$ outside;

d) $\delta(x) = \lim_{\epsilon->0} \hat{K}\epsilon^{-1}e^{-x^2/\epsilon^2}$, and find the value of $\hat{K}$.

3. *Lorentz transformations* The four vectors $p^\mu = (E, \mathbf{p})$ and $p'^\mu = (E', \mathbf{p}')$ are related by a Lorentz transformation Λ. We frequently will have to deal with integrals of the form

$$\int F(p)\, d^4p = \int F(p)\, dE\, d^3\mathbf{p}.$$

Find the 4×4 matrix representation of Λ explicitly for a boost β along the z-axis and the Jacobean for the change of variables $p^\mu \to p'^\mu$. Hence show that the 4-volume element d^4p is Lorentz invariant. Now, by making a suitable choice of $F(p)$, show that $(2E_\mathbf{p})^{-1} d^3\mathbf{p}$ is Lorentz invariant if $E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}$.

4. *Simple Harmonic Oscillator* This problem combines a review of the harmonic oscillator with that of complex coordinates. In ordinary classical mechanics, consider a two-dimensional harmonic oscillator with Lagrangian

$$L = \frac{1}{2} m (\dot{q_1}^2 + \dot{q_2}^2) - \frac{1}{2} m \omega_0^2 (q_1^2 + q_2^2),$$

where to get you in the field-theory mood I've labeled x and y as q_1 and q_2. Now rewrite these coordinates as

$$z \equiv (q_1 + iq_2)/\sqrt{2}\ , \quad z^* \equiv (q_1 - iq_2)/\sqrt{2}\ .$$

a) Find the classical equations of motion in terms of q_1 and q_2.

b) Then rederive them in terms of z and z^*, not by just plugging into the preceding, but by rewriting L in terms of z and z^*, and then minimising the action under variations $z \to z + \delta z$ and $z^* \to z^* + \delta z^*$. Treat these variations as *independent*.

c) Find the Hamiltonian in terms of z, z^* and the corresponding canonical momenta.

d) Now write the quantum Hamiltonian using lowering and raising operators defined in the usual way; a_1, $a_1^\dagger$ for q_1 and a_2, $a_2^\dagger$ for q_2. Then rewrite in terms of

$$A \equiv (a_1 + ia_2)/\sqrt{2}\ , \quad B \equiv (a_1 - ia_2)/\sqrt{2}\ .$$

e) Find the commutations relation for A, $A^\dagger$, B and $B^\dagger$ from those for a_1 etc., and check that indeed they are an independent pair of raising/lowering operators.

5. *Dirac wave equation* A plane wave solution for the Dirac equation takes the form

$$\psi \;=\; e^{-\frac{i}{\hbar}(Et - \mathbf{p}.\mathbf{x})}\psi_0$$

where

$$(\mathbf{p}.\boldsymbol{\alpha} + m\beta - E)\psi_0 = 0$$

with

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

σ_i being the Pauli sigma matrices. Show *explicitly* that the determinant condition for solutions is

$$(E^2 - (\mathbf{p}^2 + m^2))^2 = 0$$

and find the eigenspinors of definite helicity for $\mathbf{p} = (0, 0, p)$.