INTRODUCTION TO QFT 2019: PROBLEM SET 2

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Please hand in your answers at the Reception Desk of the Physics Department by 5pm on Monday Week 3 (28 October) preferably using the cover sheet on the website.

1. *Klein Gordon Field Equation I.* In this question you will do the manipulations taking us from the discretised action to the continuum field equation in the opposite order to the lecture. Start with the action for a system of coupled oscillators on a lattice of spacing a

$$S = \int_0^T dt\ a^3 \sum_k \left(\frac{1}{2} \left(\frac{\partial \phi(\mathbf{x}_k, t)}{\partial t} \right)^2 - \kappa \sum_{\hat{\mu}} \left(\phi(\mathbf{x}_{k+\hat{\mu}}, t) - \phi(\mathbf{x}_k, t) \right)^2 - \frac{1}{2}\omega^2 \phi(\mathbf{x}_k, t)^2 \right)$$

where the sites of the lattice are labelled by k and $\hat{\mu}$ runs over unit vectors connecting adjacent vertices in the *positive* x, y and z directions.

a) Find the classical equation of motion for $\phi(\mathbf{x}_k, t)$ in the discrete system.

b) Now set $\kappa = \frac{1}{2}a^{-2}$ and take the limit $a \to 0$ to obtain the Klein Gordon field equation.

2. *KG Lagrangian density.* All of these manipulations can be done purely in the continuum. Let

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi\, \partial_\mu \phi - \frac{1}{2}\omega^2 \phi^2 \tag{1}$$

be the Lagrangian Density. Consider a spatial region V_Σ bounded by the surface Σ and consider the evolution of ϕ from $\phi_0(\mathbf{x})$ at $t = 0$ to $\phi_1(\mathbf{x})$ at $t = T$ so that

$$L = \int_{V_\Sigma} d^3\mathbf{x}\, \mathcal{L}, \quad \text{and} \quad S = \int_0^T L\, dt. \tag{2}$$

a) By considering the variation $\phi(x) = \phi_{\rm cl}(x) + \delta\phi(x)$, with $\delta\phi(\mathbf{x}, 0) = \delta\phi(\mathbf{x}, T) = 0$ at the endpoints, show that

$$\delta S = \int_0^T dt \int_{V_\Sigma} d^3\mathbf{x}\, \delta\phi \left(-(\partial_0)^2 + \boldsymbol{\nabla}^2 - \omega^2 \right) \phi_{\rm cl} - \int_0^T dt \int_\Sigma \delta\phi\, \hat{\boldsymbol{n}}.\boldsymbol{\nabla}\phi_{\rm cl}\, dA \tag{3}$$

where $\hat{\boldsymbol{n}}$ is the unit normal vector to the surface Σ and dA is the element of area.

b) To deduce the KG equation we need the boundary term to vanish. Suppose Σ is the surface of a sphere of radius R; assuming that $|\delta\phi| < \textit{constant}$, how fast does $\phi_{\rm cl}$ have to fall off in order for the boundary term to vanish as $R \to \infty$? Does this cause any problem in practice?

c) Define the canonical momentum field by

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \tag{4}$$

Find an expression for the Hamiltonian Density which is defined by

$$\mathcal{H}(\pi, \phi) = \pi\, \partial_0 \phi - \mathcal{L} \tag{5}$$

(Beware, $\mathcal{H}$ can contain $\boldsymbol{\nabla}\phi$ but not $\partial_0\phi$.)

d) What are the Hamiltonian equations of motion in this case? Show that they lead to the KG equation for ϕ.

3. *Manipulating the quantum scalar field.* Given that

$$\begin{aligned}\phi(\mathbf{x}) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a^\dagger_{-\mathbf{p}}+a_\mathbf{p}\right)e^{i\mathbf{p}.\mathbf{x}}\\ \pi(\mathbf{x}) &= i\int \frac{d^3\mathbf{p}}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\left(a^\dagger_{-\mathbf{p}}-a_\mathbf{p}\right)e^{i\mathbf{p}.\mathbf{x}}\\ H &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}E_\mathbf{p}a^\dagger_\mathbf{p}a_\mathbf{p}\end{aligned}$$

a) Work backwards to find H in terms of $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$.

b) Show that

$$\mathbf{P} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\mathbf{p}\,a^\dagger_\mathbf{p}a_\mathbf{p} = -\int d^3\mathbf{x}\,\pi(\mathbf{x})\,\boldsymbol{\nabla}\phi(\mathbf{x})$$

c) By explicit calculation find the eigenvalues of $(H,\mathbf{P})$ for the two particle state $|\mathbf{p},\mathbf{p}'\rangle = a^\dagger_\mathbf{p}a^\dagger_{\mathbf{p}'}|0\rangle$.

d) Find $\langle 0|\phi(\mathbf{x})\phi(\mathbf{y})|\mathbf{p},\mathbf{p}'\rangle$.

4. *Calculating the unequal time commutator* The Green's functions that we will meet in the course can all be expressed more-or-less explicitly in terms of well-known special functions. For many purposes it is not necessary to do this but you should know how to go about it. In this question we will calculate

$$\Delta(x) = [\phi(x),\phi(0)] = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}(e^{-ip\cdot x}-e^{ip\cdot x}) \tag{6}$$

a) Set $x^\mu = (t,\mathbf{x})$ and change to polar coordinates for the momentum integral, choosing the polar axis to be along the direction of $\mathbf{x}$. Integrate out the angular variables to show that

$$\Delta(x) = \frac{-i}{8\pi^2|\mathbf{x}|}\int_{-\infty}^{\infty}\frac{\rho\,d\rho}{E(\rho)}\mathrm{Re}(e^{-i(E(\rho)t-\rho|\mathbf{x}|)}-e^{-i(E(\rho)t+\rho|\mathbf{x}|)}) \tag{7}$$

where $E(\rho)=\sqrt{\rho^2+m^2}$.

b) Now change to the *rapidity variables*: $t = s\cosh\tau$, $|\mathbf{x}| = s\sinh\tau$ and $\rho = m\sinh\phi$ (these are a very useful way of expressing kinematic quantities in a wide variety of applications). After some manipulations show that

$$\Delta(x) = \frac{-i\,m}{4\pi^2 s}\int_{-\infty}^{\infty}\cosh\phi\,(\mathrm{Re}\,e^{ims\cosh\phi})\,d\phi \tag{8}$$

What is notable about this result?

c) The remaining integral is related to the Bessel functions – find out exactly what the relationship is. If you are careful you will discover that we have made a subtle assumption in our manipulations – what is it?