Lecture 1 "Why do we need Quantum Field Theory?"

Originally, quantum mechanics formulated by Heisenberg, Schrodinger + others for non-relativistic systems.

Start with classical energy-momentum relation

$$H = \frac{\hat{p}^2}{2m} + V(x) \quad , \quad H|\psi\rangle = i\hbar \frac{\partial|\psi\rangle}{\partial t}$$

impose commutation relation

$$[p_x, x] = -i\hbar \qquad (\hbar = 0 \Rightarrow \text{classical mechanics})$$

$$[p_x, y] = [p_x, p_y] = [x, y] = 0$$

One way of doing this is to write down a wave equation

$$\left(E \to i\hbar \frac{\partial}{\partial t}\right), \quad \hat{p} \to -i\hbar \underline{\nabla}$$

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = \left(-\frac{\hbar^2}{2m} \underline{\nabla}^2 + V(x)\right) \psi(\underline{x},t)$$

which is the Schrodinger equation.** Together with the hypothesis that the expectation value of any measurable $Q(\underline{p}, \underline{x})$ is given by $\langle Q(p,x)\rangle_t = \int d^3\underline{x}\ \psi^*(\underline{x},t)\, Q(-i\hbar\underline{\nabla}, \underline{x})\, \psi(\underline{x},t)$ (where ψ are normalized to $\langle 1\rangle_t = 1$) defines q.m.

** for a single particle.

Note that with $\langle 1 \rangle_t = 1$ it is natural to interpret $\underline{\psi^* \psi}$ as the $\underline{\text{probability density}}$ for finding the particle at $\underline{x}$. There is an associated $\underline{\text{current}}$;

~~The~~ complex conjugate of Sch eqn is

$$-i\hbar \frac{\partial}{\partial t}\psi^* = \left(-\frac{\hbar^2}{2m}\underline{\nabla}^2 + V(\underline{x})\right)\psi^*$$

(assuming $V(\underline{x})$ is real); now compute

$$\psi^* \times \text{Sch. eqn} \quad - \quad \psi \times (\text{Sch. eqn})^* \quad :$$

$$i\hbar\left\{\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}\right\} = -\frac{\hbar^2}{2m}\left\{\psi^* \underline{\nabla}^2 \psi - \psi \underline{\nabla}^2 \psi^*\right\}$$

$$i\hbar \frac{\partial}{\partial t}\psi^*\psi = -\frac{\hbar^2}{2m}\Big\{\underline{\nabla}\cdot(\psi^* \underline{\nabla}\psi) - \underline{\nabla}\psi^* \cdot \underline{\nabla}\psi$$
$$- \underline{\nabla}(\psi \underline{\nabla}\psi^*) + \underline{\nabla}\psi \cdot \underline{\nabla}\psi^* \Big\}.$$

$$= -\frac{\hbar^2}{2m}\underline{\nabla}\cdot\left(\psi^* \underline{\nabla}\psi - \psi \underline{\nabla}\psi^*\right)$$

or $$\frac{\partial}{\partial t}\psi^*\psi = \cancel{\hbar}\,\underline{\nabla}\cdot\left(\frac{i\hbar}{2m}\underbrace{\psi^* \overleftrightarrow{\underline{\nabla}}\psi}\right)$$

shorthand for $\psi^* \underline{\nabla}\psi - (\underline{\nabla}\psi^*)\psi$

c.f. $$\frac{\partial \rho}{\partial t} = -\operatorname{div} \underline{j}$$

$$\rho = \psi^*\psi, \qquad \underline{j} = -\frac{i\hbar}{2m}\psi^* \overleftrightarrow{\underline{\nabla}}\psi$$

Note.

1) The probability density $\rho = \psi^* \psi$ is ~~+ve~~ definite.
" current $\underline{j} = -\frac{i\hbar}{2m}(J - J^*)$ is real

2) from $\frac{\partial \rho}{\partial t} = -\operatorname{div} \underline{j}$ we get

$$\frac{\partial}{\partial t} \int_V \rho \, d^3\underline{x} = -\int_V \operatorname{div} \underline{j} \, d^3\underline{x}$$

$$= -\int_S \underline{j} \cdot d\underline{S}$$

$= 0$ if we suppose that $\underline{j}$ vanishes sufficiently rapidly at infinity. It is therefore consistent to impose $\langle 1 \rangle_t = \int_V \rho \, d^3\underline{x} = 1$ for all t since it has no explicit time dependence.

3) ~~The et~~ An alternative way of expressing 2);
The total amount of "stuff" (as given by the probability density) is conserved. We ~~was~~ wrote down the wave equation for one particle; out of the wave eqn. we show that there is always exactly one particle.

Finally, let us look at the free particle solutions:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi$$

put $\psi = \exp -\frac{i}{\hbar}(Et - \underline{k}.\underline{x})$.

$$+E = -\frac{\hbar^2}{2m}\left(\frac{i}{\hbar}\right)^2 \underline{k}^2 = \frac{\underline{k}^2}{2m}$$

which, of course, is the classical energy-momentum relation.

However, note that $E \geqslant 0$

The energy of a quantum mechanical state is bounded below; there is a state ~~...~~ of minimum energy which cannot decay into anything else.

Many systems are <u>relativistic</u>. How is quantum mechanics generalized to deal with these? Let us try to build a wave equation for a <u>single relativistic particle</u>

$$E^2 = c^2 p^2 + m^2 c^4$$

gives us (making the usual replacements)

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = \left(-\hbar^2 c^2 \underline{\nabla}^2 + m^2 c^4\right)\psi$$

which is the <u>Klein-Gordon</u> equation. We can find the <u>probability density and current</u> in the same way as for the Schrodinger eqn.

$$-\hbar^2 \left\{ \psi^* \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi^*}{\partial t^2}\psi \right\} = -\hbar^2 c^2 \left\{ \psi^* \underline{\nabla}^2 \psi - \psi \underline{\nabla}^2 \psi^* \right\}$$

or $$\frac{\partial}{\partial t} i\left(\psi^* \overleftrightarrow{\frac{\partial}{\partial t}} \psi\right) = \underline{\nabla}\, i c^2 \left(\psi^* \overleftrightarrow{\underline{\nabla}} \psi\right)$$

so that $$\rho = i\,\psi^* \overleftrightarrow{\frac{\partial}{\partial t}} \psi$$

$$\underline{j} = \frac{c^2}{i}\, \psi^* \overleftrightarrow{\underline{\nabla}} \psi$$

Both $\underline{j}$ and ρ are real. However, <u>ρ is not +ve definite</u> and so cannot be identified as a probability density for

On the other hand $\psi^*\psi > 0$ but $\int \psi^*\psi$ is not time independent.

a single particle. / There is a difficulty in interpreting ψ as a wave function because it is impossible to define a non-negative, time independent normalization for it; we can make no statement analogous to

$$\langle Q \rangle = \int \psi^* Q \psi$$

for the K-G. equation.

We find more trouble when examining plane wave solutions

$$\psi = \exp -\frac{i}{\hbar} (Et - \underline{k}.\underline{x})$$

is a solution for $E = \pm \sqrt{k^2c^2 + m^2c^4}$

There are states of arbitrarily low energy. The system has no ground state.

INTRODUCTION TO QFT 2019

SECOND PART OF LECTURE 1: THE KLEIN GORDON EQUATION CONTINUED

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We can get a little bit more insight by considering a particle coupled to the electromagnetic field through the minimal coupling replacement

$$p^\mu \to p^\mu - eA^\mu \tag{1}$$

where e is the electric charge and $A^\mu = (V, \mathbf{A})$ is the four vector potential. With this coupling in a region of constant but non-zero potential the KG equation becomes

$$\left(i\frac{\partial}{\partial t} - eV\right)^2 \Psi = \left(-\frac{\partial^2}{\partial x^2} + m^2\right)\Psi.$$

substituting in a plane wave

$$\Psi(\mathbf{x}, t) = e^{-i(Et - \mathbf{p}.\mathbf{x})} \tag{2}$$

we get

$$(E - eV)^2 = \mathbf{p}^2 + m^2 \tag{3}$$

so

$$E = eV \pm E_\mathbf{p} \tag{4}$$

where

$$E_\mathbf{p} = +\sqrt{\mathbf{p}^2 + m^2} \tag{5}$$

It was first realised by Dirac (in the context of his wave equation, which we will come to shortly, but it also applies here) that we can interpret this as follows

(1) The '+ve' energy solution with $E = eV + E_\mathbf{p}$ describes a particle of momentum $\mathbf{p}$, kinetic energy $E_\mathbf{p}$ and electric charge e.

(2) We note that for the '-ve' energy solution we have

$$-E = -eV + E_\mathbf{p} \tag{6}$$

and interpret it as a particle of momentum $\mathbf{p}$, kinetic energy $E_\mathbf{p}$ and electric charge $-e$. That is to say, we interpret it as an *anti-particle.*

If this is right then the wave function when $V = 0$,

$$\Psi(\mathbf{x}, t) = (ae^{-iE_\mathbf{p}t} + be^{iE_\mathbf{p}t})e^{i\mathbf{p}.\mathbf{x}} \tag{7}$$

ought to be a mixture of particle and anti-particle! Compute ρ to check this out.....

$$\begin{aligned}
\rho &= \Psi^*(i\partial_t)\Psi - (i\partial_t\Psi^*)\Psi && (8)\\
&= (a^*e^{iE_\mathbf{p}t} + b^*e^{-iE_\mathbf{p}t})E_\mathbf{p}(ae^{-iE_\mathbf{p}t} - be^{iE_\mathbf{p}t}) && (9)\\
&\quad - \left(E_\mathbf{p}(-a^*e^{iE_\mathbf{p}t} + b^*e^{-iE_\mathbf{p}t})(ae^{-iE_\mathbf{p}t} + be^{iE_\mathbf{p}t})\right) && (10)\\
&= 2E_\mathbf{p}a^*a - 2E_\mathbf{p}b^*b && (11)
\end{aligned}$$

which is exactly what we would expect if ρ is actually the *charge density*!

Note that this is a purely heuristic discussion – we have not solved the problems, but we do have a pointer towards the nature of the ultimate solution which is to be found in quantum field theory.