INTRODUCTION TO QFT 2019
LECTURE 2: THE DIRAC EQUATION

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1. Basics

Historically the Dirac Equation was introduced in an attempt to rectify the defects of the KG equation, in particular the absence of a current with positive definite density, and the presence of negative energy modes. This leads to the hypothesis that the wave equation should be first order in the time derivative, while respecting the relativistic energy momentum relationship for a free particle

$$E^2 = c^2 p^2 + m^2 c^4 \tag{1}$$

We hypothesize that

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi = (c\,\boldsymbol{\alpha}.\hat{\mathbf{p}} + mc^2\beta)\psi \tag{2}$$

where we need to establish what $\boldsymbol{\alpha}, \beta$ are, and we have temporarily put hats on operators. Now for a free particle of energy E and momentum $\mathbf{p}$ we must have

$$\psi = e^{-\frac{i}{\hbar}(Et - \mathbf{p}.\mathbf{x})}\psi_0 \tag{3}$$

$$E^2\psi = \hat{H}^2\psi = (c^2(\mathbf{p}.\boldsymbol{\alpha})^2 + mc^3(\mathbf{p}.\boldsymbol{\alpha}\beta + \beta\mathbf{p}.\boldsymbol{\alpha}) + m^2c^4\beta^2)\psi \tag{4}$$

This can only reproduce (1) if

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \tag{5}$$

$$\alpha_i\beta + \beta\alpha_i = 0 \tag{6}$$

$$\beta^2 = 1 \tag{7}$$

Clearly these are not commuting objects so they must be matrices. It is easy to show that

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \tag{8}$$

where σ_i are the Pauli sigma matrices and I is the 2×2 identity matrix, satisfy these requirements. It is a good exercise to check this.

Pauli sigma matrices ...

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}, \quad \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \tag{10}$$

We have a single particle Hamiltonian H but the free particle eigenvalue equation is (from now on we revert to $\hbar = c = 1$ units)

$$(\mathbf{p}.\boldsymbol{\alpha} + m\beta - E)\psi_0 = 0 \tag{11}$$

which is four dimensional. It is easy to check that the determinant is $(E^2 - (\mathbf{p}^2 + m^2))^2$ so there are two eigenstates of positive $E = +\sqrt{\mathbf{p}^2 + m^2}$ and two of negative $E =$

$-\sqrt{\mathbf{p}^2 + m^2}$ so we have not escaped the negative energy issue. We can find the conserved charge and current starting with

$$i\frac{\partial \psi}{\partial t} = (-i\boldsymbol{\nabla}.\boldsymbol{\alpha} + m\beta)\psi \tag{12}$$

taking h.c (α, β are hermitian)

$$-i\frac{\partial \psi^\dagger}{\partial t} = (i(\boldsymbol{\nabla}\psi^\dagger).\boldsymbol{\alpha} + m\psi^\dagger\beta \tag{13}$$

multiplying the first by $\psi^\dagger$ on the left and the second by ψ on the right and subtracting gives

$$i\left(\psi^\dagger\frac{\partial \psi}{\partial t} + \frac{\partial \psi^\dagger}{\partial t}\psi\right) = -i\left(\psi^\dagger\boldsymbol{\nabla}.\boldsymbol{\alpha}\psi + (\boldsymbol{\nabla}\psi^\dagger).\boldsymbol{\alpha}\psi\right) \tag{14}$$

so $\rho = \psi^\dagger\psi$ and $\mathbf{j} = \psi^\dagger\boldsymbol{\alpha}\psi$.

2. Angular Momentum and Spin

We have the Hamiltonian

$$H = \boldsymbol{\alpha}.\mathbf{p} + m\beta \tag{15}$$

What is the angular momentum? Let us start with the orbital angular momentum

$$\boldsymbol{L} = \mathbf{x} \times \mathbf{p} \tag{16}$$

$$L_i = \epsilon_{ijk}x_j p_k \tag{17}$$

then

$$[L_i, H] = \alpha_l[\epsilon_{ijk}x_j p_k, p_l] \tag{18}$$

$$= \alpha_l\epsilon_{ijk}[x_j, p_l], p_k \tag{19}$$

$$= i\epsilon_{ijk}\alpha_j p_k \neq 0 \tag{20}$$

So we have to add something to $\boldsymbol{L}$. Consider

$$\boldsymbol{S} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{21}$$

which by construction commutes with β but not $\boldsymbol{\alpha}$. Then note

$$[\sigma_i, \sigma_j p_j] = 2\epsilon_{ijk}\sigma_k p_j \tag{22}$$

$$\tag{23}$$

it follows that

$$[S_i, H] = [S_i, \boldsymbol{\alpha}.\mathbf{p}] \tag{24}$$

$$= \frac{1}{2}2\epsilon_{ijk}\boldsymbol{\alpha}_k p_j \tag{25}$$

$$\tag{26}$$

so

$$[L_i + S_i, H] = 0 \tag{27}$$

$$\tag{28}$$

and we see that this equation describes spin-$\frac{1}{2}$ particles.

It is convenient to write the DE in a more covariant form. We use the same conventions as Peskin and Schroder. Set

$$\begin{aligned} g_{\mu\nu} &= \mathrm{diag}(1,-1,-1,-1) && (29)\\ x^{\mu} &= (t,\mathbf{x}) && (30)\\ x_{\mu} &= x^{\nu} g_{\mu\nu} = (t,-\mathbf{x}) && (31)\\ \partial^{\mu} &= \frac{\partial}{\partial x_{\mu}} = (\frac{\partial}{\partial t}, -\boldsymbol{\nabla}) && (32)\\ \partial_{\mu} &= \frac{\partial}{\partial x^{\mu}} = (\frac{\partial}{\partial t}, \boldsymbol{\nabla}) && (33)\\ p^{\mu} &= i\partial^{\mu} = (i\frac{\partial}{\partial t}, -i\boldsymbol{\nabla}) = (E,\mathbf{p}) && (34)\\ & && (35) \end{aligned}$$

Now we can rewrite the DE in this notation; multiplying through by β we get

$$\begin{aligned} 0 &= \left(-i\beta\frac{\partial}{\partial t} - i\beta\boldsymbol{\alpha}.\boldsymbol{\nabla} + m\right)\psi && (36)\\ &= \left(-i\gamma^{\mu}\partial_{\mu} + m\right)\psi && (37) \end{aligned}$$

where

$$\gamma^{\mu} = (\beta, \beta\boldsymbol{\alpha}) \tag{38}$$

are called the gamma-matrices. It is straightforward to show that

$$\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu} \tag{39}$$

by using the definition of the γs and the anticomutation properties of $(\boldsymbol{\alpha},\beta)$. In our specific form we have

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{40}$$

3. Plane Wave Solutions

For an eigenstate of 4-momentum

$$\psi = c^{-ip_{\mu}x^{\mu}}\chi(p) \tag{41}$$

substituting into the Dirac Equation we get

$$0 = \begin{pmatrix} m & -(p^0 - \mathbf{p}.\boldsymbol{\sigma}) \\ -(p^0 + \mathbf{p}.\boldsymbol{\sigma}) & m \end{pmatrix}\chi(p) \tag{42}$$

We know that the possible values of p^0 are $\pm E_{\mathbf{p}}$ where $E_{\mathbf{p}} = +\sqrt{\mathbf{p}^2 + m^2}$; note also that the spin projection operator $\mathbf{p}.\boldsymbol{S}$ commutes with the matrix. So the *eigenspinors* χ can be classified by energy and spin projection eigenvalue. [Note, the spin-projection operator is closely related to a quantity called helicity.] We let $\xi^{\pm}(\mathbf{p})$ be two-component spinors satisfying

$$\begin{aligned} \boldsymbol{\sigma}.\mathbf{p}\,\xi^{\pm}(\mathbf{p}) &= \pm|\mathbf{p}|\,\xi^{\pm}(\mathbf{p}) && (43)\\ \xi^{\pm}(\mathbf{p})^{\dagger}\xi^{\pm}(\mathbf{p}) &= 1 && (44) \end{aligned}$$

By changing $\mathbf{p} \to -\mathbf{p}$ you can see that $\xi^{\pm}(-\mathbf{p}) = \xi^{\mp}(\mathbf{p})$ We then write the four-component spinor

$$\chi(p) = \begin{pmatrix} A\,\xi \\ B\,\xi \end{pmatrix} \tag{45}$$

So for example the positive energy $p^0 = E_{\mathbf{p}}$ and positive spin projection state gives

$$0 \quad = \quad \begin{pmatrix} m & -(E_{\mathbf{p}} - |\mathbf{p}|) \\ -(E_{\mathbf{p}} + |\mathbf{p}|) & m \end{pmatrix} \begin{pmatrix} A_{++} \\ B_{++} \end{pmatrix} \tag{46}$$

where the first subscript om A means sign of the energy, and the second means sign of the spin projection. So

$$A_{++} \quad = \quad \frac{m}{E_{\mathbf{p}} + |\mathbf{p}|}\, B_{++} \tag{47}$$

$$\quad = \quad \sqrt{\frac{E_{\mathbf{p}} - |\mathbf{p}|}{E_{\mathbf{p}} + |\mathbf{p}|}}\, B_{++} \tag{48}$$

where we have used $m^2 = E_{\mathbf{p}}^2 - |\mathbf{p}|^2$. Similar exercises show

$$B_{+-} \quad = \quad \frac{m}{E_{\mathbf{p}} + |\mathbf{p}|} A_{+-} \tag{49}$$

$$B_{-+} \quad = \quad -\frac{m}{E_{\mathbf{p}} + |\mathbf{p}|} A_{-+} \tag{50}$$

$$A_{--} \quad = \quad -\frac{m}{E_{\mathbf{p}} + |\mathbf{p}|} B_{--} \tag{51}$$

$$\tag{52}$$

The last thing to fix is the normalization. Remember that $\psi^\dagger\psi$ is a density with corresponding current $\mathbf{j}$ so we expect it to be the 0-component of a 4-vector. The standard normalization is

$$\chi^\dagger\chi = 2E_{\mathbf{p}} \tag{53}$$

The $E_{\mathbf{p}}$ is exactly the 0-component of a 4-vector, the 2 is purely convention. With this normalization we obtain

(1) the positive energy spinors

$$u^{\pm}(\mathbf{p}) = \begin{pmatrix} \sqrt{E_{\mathbf{p}} \mp |\mathbf{p}|}\,\xi^{\pm}(\mathbf{p}) \\ \sqrt{E_{\mathbf{p}} \pm |\mathbf{p}|}\,\xi^{\pm}(\mathbf{p}) \end{pmatrix} \tag{54}$$

(2) and the negative energy spinors

$$v^{\pm}(-\mathbf{p}) = \begin{pmatrix} \sqrt{E_{\mathbf{p}} \pm |\mathbf{p}|}\,\xi^{\pm}(\mathbf{p}) \\ -\sqrt{E_{\mathbf{p}} \mp |\mathbf{p}|}\,\xi^{\pm}(\mathbf{p}) \end{pmatrix} \tag{55}$$

The minus sign in the definition of the negative energy spinors is deliberate – the reason for it will become apparent when we construct the quantum field theory for Dirac particles.
Exercises

(1) show using (14) that $\partial_\mu j^\mu$=0 with $j^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi$.

(2) show that all the different spinors are orthogonal

4. Properties under Lorentz Transformations

We have seen that $\psi^\dagger\psi$ is not a Lorentz scalar. So what combination *is* a scalar? The answer is $\psi^\dagger\gamma^0\psi$. Computing explicitly for the positive energy spinors we get

$$u^\pm(\mathbf{p})^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} u^\pm(\mathbf{p}) = 2 * \sqrt{E_\mathbf{p} \mp |\mathbf{p}|}\sqrt{E_\mathbf{p} \pm |\mathbf{p}|} \tag{56}$$

$$= 2m \tag{57}$$

Similarly we find that

$$v^\pm(\mathbf{p})^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} v^\pm(\mathbf{p}) = -2 * \sqrt{E_\mathbf{p} \mp |\mathbf{p}|}\sqrt{E_\mathbf{p} \pm |\mathbf{p}|} \tag{58}$$

$$= -2m \tag{59}$$

Of course this is explicitly a Lorentz scalar.

5. Final Comments

We will need all this apparatus later but the take-home messages are

(1) The negative energy states are still there

(2) They describe anti-particles just as in the KG case

(3) Relativistic QM is bound to fail because actually particles and antiparticles can annihilate so the number of particles is not constant – and the formalism of a wave function for a fixed number of particles cannot cope with this.

(4) So we need Quantum Field Theory which will be able to handle varying particle number.

(5) Relativistic systems are not the only variable-number systems. We now understand that many quasi-particle descriptions of CM systems are also like this and can also be described by quantum field theories.