Lecture 3 The Formalism of Classical Field Theory

We'll start with a quick review of Lagrangian and Hamiltonian mechanics for a single particle. If this material is new to you, you cannot learn it from these notes. The standard work on the topic is "Classical Mechanics" by Goldstein – Chapters 1, 2 and 8 in the 3rd edition.

For a particle of mass m, position $\underline{x}(t)$ moving in a potential $V(\underline{x})$ we define the Lagrangian

$$L = \text{Kinetic Energy} - \text{Potential Energy}$$
$$= \frac{1}{2} m \dot{\underline{x}}^2 - V(\underline{x}) \qquad (1)$$

For the trajectory

we define the action

$$S = \int_{t=0}^{T} L(\dot{\underline{x}}(t), \underline{x}(t))\, dt$$

The Principal of Least Action states that

" The actual path followed by the particle $\hat{\underline{x}}(t)$ is such that S is an extremum "

So we write

$$\underline{x}(t) = \hat{\underline{x}}(t) + \delta \underline{x}(t)$$

where $\delta \underline{x}(0) = \delta \underline{x}(T) = 0$ so the trajectory definitely starts at $\underline{x}_0$ at $t=0$ and ends at $\underline{x}_1$ at $t=T$.

Then

$$S(\underline{x}(t)) = \int_0^T L\left(\dot{\hat{\underline{x}}}(t) + \delta\dot{\underline{x}}(t),\ \hat{\underline{x}}(t) + \delta\underline{x}(t)\right) dt$$

$$= S(\hat{\underline{x}}(t)) + \int_0^T \frac{\partial L}{\partial \dot{x}_i}\delta\dot{x}_i + \frac{\partial L}{\partial x_i}\delta x_i \; dt$$

(with summation over repeated indices). Integrating the first term by parts

0 by boundary conditions

$$= S(\hat{\underline{x}}(t)) + \cancel{\left[\frac{\partial L}{\partial \dot{x}_i}\delta x_i\right]_0^T}$$

$$+ \int_0^T \left(-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \frac{\partial L}{\partial x_i}\right)\delta x_i \; dt$$

The PLA then implies that $\hat{\underline{x}}(t)$ is a solution of the Lagrange equations

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \frac{\partial L}{\partial x_i} = 0$$

For our Lagrangian (1) this gives

$$m\ddot{\underline{x}} = -\underline{\nabla} V$$

which is Newton's Law.

The quantity $\dfrac{\partial L}{\partial \dot{x}_i}$ is defined to be the <u>canonical momentum</u> conjugate to x_i. We call it

$$p_i = \frac{\partial L}{\partial \dot{x}_i}$$

In the case of a single particle in a potential it is of course the same as the usual momentum.

(p_i, x_i) are called a canonical coordinate pair.

We can invert the equation(s) $p_i = \frac{\partial L}{\partial \dot{x}_i}$

to get $\dot{x}_i(\underline{p}, \underline{x})$ and define the Hamiltonian

$$H(\underline{p}, \underline{x}) = p_i \, \dot{x}_i(\underline{p}, \underline{x}) - L(\underline{\dot{x}}(\underline{p}, \underline{x}), \underline{x})$$

$$= \frac{\underline{p}^2}{2m} + V(\underline{x})$$

in our single particle example.

As you know the Hamiltonian plays a crucial role in quantum mechanics.

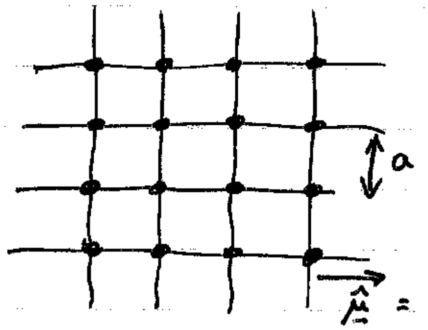
In classical mechanics the Hamiltonian equations of motion are

$$\frac{\partial H}{\partial x_i} = -\dot{p}_i \, , \quad \frac{\partial H}{\partial p_i} = \dot{x}_i$$

There is an exercise in these manipulations in Problem Set 1.

Moving on to fields

We'll start by considering a lattice ~~of points~~



vertices have coordinates

$$\underline{x}_k = (x_k, y_k, z_k) = a(k_1 \hat{\underline{x}} + k_2 \hat{\underline{y}} + k_3 \hat{\underline{z}})$$

$k_{1,2,3}$ integers

$\hat{\mu} = \hat{\underline{x}}, \hat{\underline{y}}, \hat{\underline{z}}$ unit vectors

At each vertex put a simple harmonic oscillator degree of freedom $\phi(t, \underline{x}_k)$ - you can think literally of a spring with displacement from equilibrium ϕ. The Lagrangian for the oscillator at site k is

$$L_k = \frac{1}{2}\left(\frac{\partial \phi(t, \underline{x}_k)}{\partial t}\right)^2 - \frac{1}{2}\omega^2 \phi(t, \underline{x}_k)^2$$

so this oscillator has unit mass and frequency ω

The Lagrangian for all the oscillators is

$$L = a^3 \sum_k L_k$$

where we introduced the factor a^3 for later convenience - note that being an overall factor it does not affect the equations of motion.

So far the oscillators are independent. We can couple them together by introducing

$$V_{int} = K \sum_{k,\hat{\underline{\mu}}} \left(\phi(t, \underline{x}_k + a\hat{\underline{\mu}}) - \phi(t, \underline{x}_k) \right)^2$$

with $K > 0$. This encourages the oscillators to move coherently with their neighbours. So now the action for the coupled system is

$$S = \int_0^T dt \, a^3 \sum_k \left\{ \frac{1}{2} \left(\frac{\partial \phi(t, \underline{x}_k)}{\partial t} \right)^2 - K \sum_{\hat{\underline{\mu}}} \left(\phi(t, \underline{x}_k + a\hat{\underline{\mu}}) - \phi(t, \underline{x}_k) \right)^2 - \frac{1}{2} \omega^2 \phi(t, \underline{x}_k)^2 \right\}$$

We can recover a continuous space by taking $a \to 0$. Later we will do this carefully but for now note that

$$\phi(t, \underline{x}_k + a\hat{\underline{\mu}}) - \phi(t, \underline{x}_k) = a \frac{\partial \phi(t, \underline{x})}{\partial x_\mu} + O(a^2) \qquad \mu = 1, 2, 3$$

and $a^3 \sum_k \to \int_V d^3\underline{x}$. Choosing $K = \frac{1}{2a^2}$ we find

$$S = \int_0^T dt \int_V d^3\underline{x} \left\{ \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left(\nabla \phi \right)^2 - \frac{1}{2} \omega^2 \phi^2 \right\}$$

where $\phi = \phi(t, \underline{x})$ is a field defined on a continuous space $\underline{x}$.

we denote the Lagrangian density

$$\mathcal{L}\left(\frac{\partial\phi}{\partial t}, \underline{\nabla}\phi, \phi\right) = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}(\underline{\nabla}\phi)^2 - \tfrac{1}{2}\omega^2\phi^2$$

$$= \mathcal{L}(\partial_\mu\phi, \phi) \qquad \mu = 0,1,2,3$$

$$= \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}\omega^2\phi^2$$

The PLA now says that the actual field configuration $\bar{\phi}(t,\underline{x})$ is an extremum of S so put $\phi = \bar{\phi} + \delta\phi$ then

$$S = \int_0^T dt \int_V d^3x \left\{ \frac{1}{2}\left(\frac{\partial\bar{\phi}}{\partial t} + \frac{\partial\delta\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\underline{\nabla}\phi + \underline{\nabla}\delta\phi\right)^2 - \tfrac{1}{2}\omega^2(\bar{\phi}+\delta\phi)^2 \right\}$$

$$= \bar{S} + \int_0^T dt \int_V d^3x \left\{ \frac{\partial\bar{\phi}}{\partial t}\frac{\partial\delta\phi}{\partial t} - \underline{\nabla}\phi\cdot\underline{\nabla}\delta\phi - \omega^2\bar{\phi}\,\delta\phi \right\}$$

we can integrate both $\frac{\partial}{\partial t}$ and $\underline{\nabla}$ by parts to get

$$= \bar{S} + \underbrace{\int_V d^3x \left[\frac{\partial\bar{\phi}}{\partial t}\delta\phi\right]_0^T}_{=0} - \underbrace{\int_0^T\int_{\partial V} d\underline{S}\cdot\underline{\nabla}\phi\;\delta\phi}_{=0 \text{ if } \underline{\nabla}\phi \text{ vanishes fast enough at } \infty}$$

$$+ \int_0^T\int_V d^3x \left\{ -\frac{\partial^2\bar{\phi}}{\partial t^2} + \underline{\nabla}^2\bar{\phi} - \omega^2\bar{\phi} \right\}\delta\phi$$

which gives us the <u>field equation</u>

$$\frac{\partial^2\phi}{\partial t^2} - \underline{\nabla}^2\phi + \omega^2\phi = 0.$$

or $\partial_\mu \partial^\mu \phi + \omega^2\phi = 0$

which is the Klein Gordon equation again !!
But this time it is a field equation, <u>not</u> a wave equation.
You can derive the same equation by working with the discrete version, finding the e.o.m., and then taking $a \to 0$. That is in Problem Set 2

<u>Quantizing the system</u>

Since we don't know how to quantize a field yet we now go back to the oscillator system and quantize that. To save some writing we will define

$$\phi_k = \phi(t, x_k), \quad \dot{\phi}_k = \frac{\partial}{\partial t}\phi(t, x_k)$$

so

$$L = a^3 \sum_k \left(\frac{1}{2}(\dot{\phi}_k)^2 - \kappa \sum_{\hat{\mu}} (\phi_{k+\hat{\mu}} - \phi_k)^2 - \frac{1}{2}\omega^2 \phi_k^2 \right)$$

To quantize we need the Hamiltonian so

the momentum π_k conjugate to ϕ_k is

$$p_k = \frac{\partial L}{\partial \dot\phi_k} = a^3 \dot\phi_k = a^3 \pi_k$$

and the Hamiltonian is

$$H = \sum_k p_k \dot\phi_k - L$$

$$= \sum_k a^3 \pi_k^2$$

$$- a^3 \sum_k \left(\frac{1}{2} \pi_k^2 - \kappa \sum_{\hat\mu} (\phi_{k+\hat\mu} - \phi_k)^2 - \tfrac{1}{2} \omega^2 \phi_k^2 \right)$$

$$= a^3 \sum_k \left(\frac{1}{2} \pi_k^2 + \kappa \sum_{\hat\mu} (\phi_{k+\hat\mu} - \phi_k)^2 + \frac{1}{2} \omega^2 \phi_k^2 \right)$$

Quantizing we should have by the postulates of Q.M.

$$[p_k, \phi_\ell] = -i\hbar\, \delta_{k\ell}, \quad \text{so} \quad [\pi_k, \phi_\ell] = -\frac{i\hbar\, \delta_{k\ell}}{a^3}$$

and all others commute $[\pi_k, \pi_\ell] = [\phi_k, \phi_\ell] = 0$

Note that

$$H = a^3 \sum_k H_k$$

and if we take $a \to 0$, $\kappa = \frac{1}{2a^2}$ we get

$$H = \int d^3\underline{x}\, \mathcal{H}$$

where $\mathcal{H}$ is the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\,\pi(t,\underline{x})^2 + \frac{1}{2}\left(\underline{\nabla}\,\phi(t,\underline{x})\right)^2 + \frac{1}{2}\,\omega^2\,\phi(t,\underline{x})^2$$