Lecture 5 The Scalar QFT in the Continuum

We can start with the scalar field theory on the lattice that we developed in the previous lecture and use it to construct a continuum scalar QFT.

There are two stages:

1 Take $N \to \infty$, $a \to 0$ in such a way that $L = Na$ remains fixed. This gives us a continuum theory in a box of size L.

2 Take $L \to \infty$. This is the infinite volume, or thermodynamic, limit.

Taking $a \to 0$, $N \to \infty$

$$a^3 \sum_{\underline{x} \in \chi_3} \to \int_V d^3 \underline{x}$$

where V is a box of side L.

as $a \to 0$ this becomes an ∞ spike at $\underline{x} = \underline{y}$ so we expect it to be a $\delta^3(\underline{x} - \underline{y})$ but we need to determine the coefficient

Remember that $[\pi(\underline{x}), \phi(\underline{y})] = -\frac{i}{a^3} \delta_{\underline{x},\underline{y}}$

so $a^3 \sum_{\underline{x} \in \chi_3} [\pi(\underline{x}), \phi(\underline{y})] = -i$

which becomes

$$\int_V d^3x \, [\pi(\underline{x}), \phi(\underline{y})] = -i$$

from which we deduce that

$$[\pi(\underline{x}), \phi(\underline{y})] = -i\, \delta^3(\underline{x}-\underline{y})$$

Note that this conclusion is unchanged if we then take $L \to \infty$.

A unit cell in momentum space has volume $\left(\frac{2\pi}{L}\right)^3$

so $$\left(\frac{2\pi}{L}\right)^3 \sum_{p \in B_3} \longrightarrow \int d^3 p$$

Remember that
$$\phi(\underline{x}) = C^3 \sum_{\underline{p} \in B_3} \bar{\Phi}(\underline{p}) e^{i \underline{p}\cdot\underline{x}}$$

so we choose $C = L^{-1}$ so then ~~thus~~

$$\phi(\underline{x}) = \frac{1}{(2\pi)^3} \left(\frac{2\pi}{L}\right)^3 \sum_{\underline{p} \in B_3} \bar{\Phi}(\underline{p}) e^{i\underline{p}\cdot\underline{x}}$$
$$= \int \frac{d^3 \underline{p}}{(2\pi)^3} \bar{\Phi}(\underline{p}) e^{i\underline{p}\cdot\underline{x}}$$

when $L \to \infty$.

We then have also that, since $\bar{L} = C^2 L = L^{-1}$,

$$[\bar{\pi}(\underline{p}), \bar{\Phi}(\underline{q})] = -i\, L^3\, \delta_{\underline{p}+\underline{q}, 0}$$ this time as $L \to \infty$ we expect r.h.s. to be a δ fn.

so $$\left(\frac{2\pi}{L}\right)^3 \sum_{\underline{p} \in B_3} [\bar{\pi}(\underline{p}), \bar{\Phi}(\underline{q})] = -i(2\pi)^3$$

and $$[\bar{\pi}(\underline{p}), \bar{\Phi}(\underline{q})] = -i(2\pi)^3 \delta^3(\underline{p}+\underline{q})$$

Now we define the annihilation and creation operators without the $\bar{L}^{3/2}$ factors that we had previously

$$a_p = i \frac{1}{\sqrt{2E_p}} \left(\Pi(p) - i E_p \Phi(p) \right)$$

$$a_p^+ = -i \frac{1}{\sqrt{2E_p}} \left(\Pi(-p) + i E_p \Phi(-p) \right)$$

Then, ignoring the vacuum contribution we have

$$H = \tilde{L}^3 \sum_{p \in B_3} E_p \, a_p^+ \, a_p$$

$$= \frac{1}{L^3} \sum_{p \in B_3} E_p \, a_p^+ \, a_p$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p \, a_p^+ \, a_p$$

where, provided we consider only finite momentum modes, and set $K = \frac{1}{2a^2}$

$$E_p = +\sqrt{\omega^2 + \frac{1}{2a^2} \cdot 8 \cdot \left(\frac{a^2 p_x^2 + a^2 p_y^2 + a^2 p_z^2}{4} \right) + O(a^2)}$$

$$\to \sqrt{\omega^2 + p^2}$$

ie the energy of a relativistic particle of mass ω and momentum p.

Finally note that we have

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{-p}^+ + a_p \right) e^{ip \cdot x}$$

$$\pi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \times -i \quad e^{i\underline{p}\cdot\underline{x}} \times (a_{\underline{p}} - a^{+}_{-\underline{p}})$$

$$[a^{+}_{\underline{p}} \;, a_{\underline{q}}] = -(2\pi)^3 \delta^3(\underline{p}-\underline{q})$$

These formulae define scalar QFT in the continuum. The formalism describes a collection of <u>relativistic particles</u> in which the <u>number</u> of particles is <u>variable</u> and we have banished the problem of -ve energy states. ~~$a^{+}_{\underline{p}}$ creates~~ $a^{+}_{\underline{p}}$ adds a ~~state~~ particle of three momentum $\underline{p}$ and energy E_p to the system: Suppose

$$H|\psi\rangle = E|\psi\rangle$$

then

$$H a^{+}_{\underline{q}} |\psi\rangle = \int \frac{d^3p}{(2\pi)^3} E_p \, a^{+}_{\underline{p}} \, a_{\underline{p}} \, a^{+}_{\underline{q}} |\psi\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p \, a^{+}_{\underline{p}} \left(a^{+}_{\underline{q}} a_{\underline{p}} + (2\pi)^3 \delta^3(\underline{p}-\underline{q}) \right) |\psi\rangle$$

$$= a^{+}_{\underline{q}} (H + E_{\underline{q}}) |\psi\rangle$$

$$= (E + E_{\underline{q}}) \, a^{+}_{\underline{q}} |\psi\rangle$$

Similarly $a_{\underline{p}}$ removes a particle of three momentum $\underline{p}$ and energy $E_{\underline{p}}$ from the system.

As well as the Hamiltonian we can define a number operator

$$N = \int \frac{d^3 p}{(2\pi)^3} \, a^{+}_{\underline{p}} a_{\underline{p}}$$

so that if $N|\psi\rangle = N_0|\psi\rangle$ then

$$N a^{+}_{\underline{p}} |\psi\rangle = (N_0 + 1) \, a^{+}_{\underline{p}} |\psi\rangle$$

and $N a_{\underline{p}} |\psi\rangle = (N_0 - 1) \, a_{\underline{p}} |\psi\rangle$ provided of course that $a_{\underline{p}}|\psi\rangle$ is not zero.

We can also define a three momentum operator $\underline{P}$ (NB this is different from the canonical momentum π!)

$$\underline{P} = \int \frac{d^3 p}{(2\pi)^3} \, \underline{p} \, a^{+}_{\underline{p}} a_{\underline{p}}$$

so that if $\underline{P}|\psi\rangle = \underline{P}_0|\psi\rangle$ then

$$\underline{P} \, a^{+}_{\underline{p}} |\psi\rangle = (\underline{P}_0 + \underline{p}) \, a^{+}_{\underline{p}} |\psi\rangle \quad \text{etc.}$$

<u>Operator ordering</u>

You would be quite correct to worry about the order of a^{+} and a in H etc and about the discarding of the (infinite) vacuum energy term.

The ambiguity is dealt with by introducing

<u>Normal Ordering</u>

$$: \phi(x_1) \ldots \phi(x_n) :$$

is the product written out with all creation operators to the left of all annihilation operators. Then

$$H = : \int d^3\underline{x} \; \frac{1}{2}\left(\pi(\underline{x})^2 + (\underline{\nabla}\,\phi(\underline{x}))^2 + \omega^2 \phi(\underline{x})^2\right) :$$

$$= \int \frac{d^3p}{(2\pi)^3} \; E_p \; a_p^{+} \; a_p$$

But there is a point of principle. If we have real degrees of freedom that couple directly to energy density then the absolute value matters and we are not free to redefine the zero point. This is a problem when we consider gravity – but in this course we will ~~deal with co~~ stick to a gravity-free world.

Normalization of states

It is natural to regard the vacuum state as normalized to 1

$$\langle 0 | 0 \rangle = 1$$

We define a single particle state

$$|\underline{p}\rangle = \sqrt{2E_{\underline{p}}}\; a^{\dagger}_{\underline{p}} \;|0\rangle$$

so that

$$\langle \underline{q} | \underline{p} \rangle = \sqrt{2E_{\underline{p}}}\sqrt{2E_{\underline{q}}} \;\langle 0 | a_{\underline{q}}\, a^{\dagger}_{\underline{p}} \,|0\rangle$$

$$= \sqrt{2E_{\underline{p}}}\sqrt{2E_{\underline{q}}} \;\langle 0 | a^{\dagger}_{\underline{p}}\, a_{\underline{q}} + (2\pi)^3 \delta^3(\underline{p}-\underline{q}) \,|0\rangle$$

$$= (2\pi)^3\, 2E_{\underline{p}}\; \delta^3(\underline{p}-\underline{q})$$

which is the orthogonality condition for single particle states. Note that

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\underline{p}}} \langle \underline{q} | \underline{p} \rangle \langle \underline{p} | \underline{q}' \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\underline{p}}} \;(2\pi)^3\, 2E_{\underline{p}}\; \delta^3(\underline{p}-\underline{q}) \times (2\pi)^3\, 2E_{\underline{p}}\; \delta^3(\underline{p}-\underline{q}')$$

$$= (2\pi)^3\, 2E_{\underline{p}}\; \delta^3(\underline{q}-\underline{q}')$$

so $\quad (\mathbb{1})_{\text{single particle}} = \displaystyle\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\underline{p}}} \;|\underline{p}\rangle\langle \underline{p}|$

The measure $\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}$ is Lorentz invariant (see Problem Set 1)

We can also construct the wave function for a single particle in position space by

$\langle 0 | \phi(\underline{x}) | \underline{p} \rangle$

$$= \langle 0| \int \frac{d^3\underline{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\underline{q}}}} e^{i\underline{q}\cdot\underline{x}} \left(a^\dagger_{-\underline{q}} + a_{\underline{q}} \right) \sqrt{2E_p}\, a^\dagger_{\underline{p}}(p) |0\rangle .$$

$$= \int \frac{d^3\underline{q}}{(2\pi)^3} e^{i\underline{q}\cdot\underline{x}} \frac{\sqrt{2E_p}}{\sqrt{2E_{\underline{q}}}} \langle 0 | a_{\underline{q}} \; a^\dagger_{\underline{p}} | 0 \rangle$$

$$= \int \frac{d^3\underline{q}}{(2\pi)^3} e^{i\underline{q}\cdot\underline{x}} \frac{\sqrt{2E_p}}{\sqrt{2E_{\underline{q}}}} (2\pi)^3 \delta^3(\underline{p}-\underline{q})$$

$$= e^{i\underline{p}\cdot\underline{x}}$$

which is the equivalent of $\langle \underline{x} | \underline{p} \rangle$ in ordinary quantum mechanics.

We can extend these considerations to the n-particle sector without much difficulty.