Lecture 6 Introducing Time

So far we have developed the time independent properties of the scalar field. To examine the time dependent properties we will work in the Heisenberg Picture. Just to remind you, in this picture

states $| \quad \rangle$ are time-independent

but

operators Q satisfy

$$i \frac{dQ}{dt} = [Q, H] \qquad (*)$$

In quantum mechanics it is straightforward to show that this is equivalent to the Schrodinger Picture in which the wavefunction carries the time dependence and operators are time independent.

The formal solution to (*) is

$$Q(t) = e^{iHt} Q(0) e^{-iHt}$$

as you can easily check by expanding e^{iHt} that $\frac{d}{dt} e^{iHt} = i e^{iHt} H$ and the h.c.

Let's examine the time evolution of H, $\underline{P}$, N, ϕ and π.

1. Clearly $\frac{dH}{dt} = 0$ as $[H, H] = 0$, so the total energy is conserved.

To calculate the others we will need our basic commutator

$$[a_{\underline{p}}^+, a_{\underline{q}}] = -(2\pi)^3 \delta^3(\underline{p} - \underline{q})$$

plus

$$\begin{aligned}[a_{\underline{p}}^+ a_{\underline{p}}, a_{\underline{q}}] &= a_{\underline{p}}^+ a_{\underline{p}} a_{\underline{q}} - a_{\underline{q}} a_{\underline{p}}^+ a_{\underline{p}} \\ &= a_{\underline{p}}^+ a_{\underline{p}} a_{\underline{q}} - \left(a_{\underline{p}}^+ a_{\underline{q}} + (2\pi)^3 \delta^3(\underline{p} - \underline{q})\right) \times a_{\underline{p}} \\ &= -(2\pi)^3 \delta^3(\underline{p} - \underline{q})\, a_{\underline{p}}\end{aligned}$$

and

$$[a_{\underline{p}}^+ a_{\underline{p}}, a_{\underline{q}}^+ a_{\underline{q}}] = a_{\underline{p}}^+ a_{\underline{p}} a_{\underline{q}}^+ a_{\underline{q}} - a_{\underline{q}}^+ a_{\underline{q}} a_{\underline{p}}^+ a_{\underline{p}}$$

$$= a_{\underline{p}}^{+} \left[a_{\underline{p}}, a_{\underline{q}}^{+} a_{\underline{q}} \right]$$
$$+ \left[a_{\underline{p}}^{+}, a_{\underline{q}}^{+} a_{\underline{q}} \right] a_{\underline{p}}$$
$$= \left(a_{\underline{p}}^{+} a_{\underline{q}} - a_{\underline{q}}^{+} a_{\underline{p}} \right) (2\pi)^3 \delta^3(\underline{p}-\underline{q})$$
$$= 0$$

We see immediately that

2. $$\frac{dN}{dt} = 0 \quad \text{and} \quad \frac{d\underline{P}}{dt} = 0$$

so the particle number and system three-momentum are conserved (remember that at the moment our particles do not interact with each other so this is what we would expect).

To find $\frac{\partial \phi}{\partial t}$ and $\frac{\partial \pi}{\partial t}$ we need

$$f^{\pm} = e^{iHt} \begin{Bmatrix} a_{\underline{p}}^{+} \\ a_{\underline{p}} \end{Bmatrix} e^{-iHt}$$

Note that

$$a_{\underline{p}}^{+} H = a_{\underline{p}}^{+} \int \frac{d^3 q}{(2\pi)^3} E_{\underline{q}} a_{\underline{q}}^{+} a_{\underline{q}}$$
$$= \int \frac{d^3\underline{q}}{(2\pi)^3} E_{\underline{q}} \left(a_{\underline{q}}^{+} a_{\underline{q}} a_{\underline{p}}^{+} - (2\pi)^3 \delta^3(\underline{p}-\underline{q}) a_{\underline{p}}^{+} \right)$$

$$= (H - E_p)\, a_p^+$$

so $$f^+ = e^{iHt}\, e^{-i(H-E_p)t}\, a_p^+$$

$$= e^{iE_pt}\, a_p^+ \qquad \text{because } [H, H-E_p] = 0$$

similarly $$f^- = e^{-iE_pt}\, a_p$$

Remember that we found that at fixed $t\,(=0)$

$$\phi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{-p}^+ + a_p \right) e^{i\underline{p}\cdot\underline{x}}$$

so

$$\phi(t,\underline{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(e^{iE_pt} a_{-p}^+ + e^{-iE_pt} a_p \right) \times e^{i\underline{p}\cdot\underline{x}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(e^{i(E_pt - \underline{p}\cdot\underline{x})} a_p^+ + e^{-i(E_pt - \underline{p}\cdot\underline{x})} a_p \right)$$

where we changed integration variable $\underline{p} \to -\underline{p}$ in the first term

and

$$\pi(\underline{x}) = i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(a_{-p}^+ - a_p \right) e^{i\underline{p}\cdot\underline{x}}$$

so

$$\pi(t,\underline{x}) = i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(e^{iE_pt} a_{-p}^+ - e^{-iE_pt} a_p \right) \times e^{i\underline{p}\cdot\underline{x}}$$

$$= i \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{\sqrt{2E_p}} \left(e^{i(E_pt - \underline{p}\cdot\underline{x})} a_p^+ - e^{-i(E_pt - \underline{p}\cdot\underline{x})} a_p \right)$$

$$= \frac{\partial}{\partial t} \phi(t, \underline{x})$$

Things to Note

1. ~~So~~ The quantum field operators have exactly the same relationship between ~~I~~ field and conjugate momentum field as in classical field theory.

2. +ve frequencies $\longleftrightarrow$ annihilation operator a_k

   -ve frequencies $\longleftrightarrow$ creation operator a_k^+

3. π and ϕ both satisfy the Klein Gordon equation

$$(\Box + m^2)\,\phi(t, \underline{x}) = 0.$$

from now on we will call masses "m" not "ω"

This is true as an operator statement. As you learn more QFT you will discover that sometimes we only have a weaker version eg.

$$(\Box + m^2)\,\phi(t, x)\,|\,\text{state}\rangle = 0$$

Causality and Unequal Time Commutators

We can write

$$\phi(x=(t,\underline{x})) = \int \frac{d^3\underline{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\underline{p}}}} \left(e^{ip\cdot x} a^+_{\underline{p}} + e^{-ip\cdot x} a_{\underline{p}} \right)$$

where $p = (E_{\underline{p}}, \underline{p})$. Then

$$[\phi(x), \phi(y)] = \int \frac{d^3\underline{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\underline{p}}}} \int \frac{d^3\underline{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\underline{q}}}} \times$$

$$\left(e^{ip\cdot x} e^{-iq\cdot y} [a^+_{\underline{p}}, a_{\underline{q}}] + e^{-ip\cdot x} e^{iq\cdot y} [a_{\underline{p}}, a^+_{\underline{q}}] \right).$$

$$= \int \frac{d^3\underline{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\underline{p}}}} \int \frac{d^3\underline{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\underline{q}}}} \times (2\pi)^3 \delta^3(\underline{p}-\underline{q}) \times$$

$$\left(-e^{i(p\cdot x - q\cdot y)} + e^{-i(p\cdot x - q\cdot y)} \right)$$

$$= \int \frac{d^3\underline{p}}{(2\pi)^3} \frac{1}{2E_{\underline{p}}} \left(-e^{ip\cdot(x-y)} + e^{-ip\cdot(x-y)} \right)$$

$$\equiv \Delta(x-y)$$

What are the properties of Δ? We expect that when the interval $x-y$ is space-like $(x-y)^2<0$ $\Delta(x-y)=0$ because no signal can causally connect what happen at x, y. Conversely if the interval is time-like then we expect $\Delta(x-y) \neq 0$.

1. If $x-y$ is <u>space-like</u> choose to work in a frame where $(x-y)_0 = 0$; we can do this because $\Delta(x-y)$ is manifestly Lorentz invariant. Then

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(-e^{-i\,\underline{p}\cdot(\underline{x}-\underline{y})} + e^{i\,\underline{p}\cdot(\underline{x}-\underline{y})} \right).$$

changing variable $\underline{p} \to -\underline{p}$ in the second term we get zero.

2. If $x-y$ is <u>time-like</u> then we can look at the special case $\underline{x} = \underline{y}$. Then $(t = (x-y)_0)$

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{iE_p t} - e^{-iE_p t} \right)$$

which doesn't vanish by any change of variables! The integral can be done explicitly. This is an exercise in the Problem Set 2.

If you have done Problem Set 1 you might expect that Δ is related to a four dimensional integral and that is indeed the case.

Let's consider

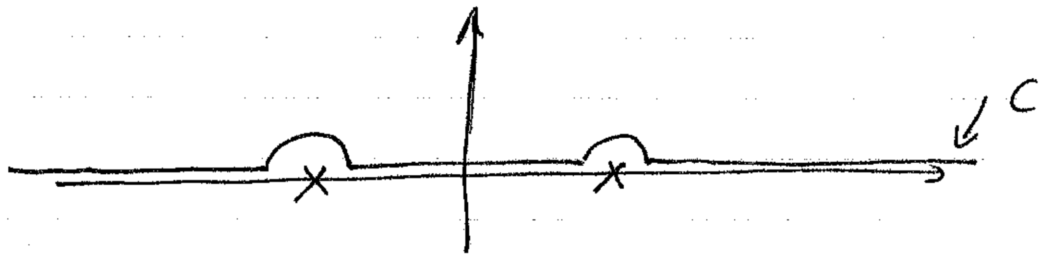
$$D(x) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip\cdot x}$$

$$= \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{i}{p_0^{\,2}-(\underline{p}^2+m^2)} e^{-i(p_0t-\underline{p}\cdot\underline{x})}$$

This is ill-defined as it stands because of the poles at

$$p_0 = \pm\sqrt{\underline{p}^2+m^2}$$

so we have to specify the contour. Choose



then $D(x) = \int \frac{d^3p}{(2\pi)^3} e^{i\underline{p}\cdot\underline{x}} \int_C \frac{dp_0}{2\pi} \frac{i}{p_0^2-(\underline{p}^2+m^2)} e^{-ip_0t}$

is well defined.

If $t<0$ then we can close the contour in the upper half plane. By the Jordan Lemma the integral over the semi-circle is zero. No poles are enclosed so

$$D(x) = 0$$

If $t>0$ we must close in the lower half plane. This time two poles are enclosed so

$$D(x) = \Theta(t) \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} \times 2\pi i \times \frac{i}{2\pi} \times$$
$$\left(\frac{1}{2E_p} e^{-iE_p t} - \frac{1}{2E_p} e^{iE_p t} \right)$$

$$= \Theta(t) \times \Delta(x)$$

$D(x)$ is a Green's Function of the KG operator as we can see from

$$(\Box + m^2)\, D(x) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} (m^2 - p^2)\, e^{-ip\cdot x}$$
$$= -i \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}$$
$$= -i\,\delta^4(x)$$

so $D(x)$ is in fact the <u>retarded Green's Function</u>, which describes classical causal propagation, we will re-label it $D_R(x)$ to reflect that.

There is more to be said on the matter of Green's Functions. We'll follow that up in a couple of weeks time.