INTRODUCTION TO QFT 2019: PROBLEM SET 3

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Please hand in your answers at the Reception Desk of the Physics Department by 5pm on *Wednesday* Week 5 (13 November) preferably using the cover sheet on the website. Your class tutor may have given you different instructions for hand-in times – in that case please do as they ask.

1. *Particle Projection Operators.* By extending the 1-particle projection operator that we derived in the lectures show that the n-particle projection operator for a real scalar field is

$$P_n = \frac{1}{n!}\prod_{i=1}^{n}\left(\int \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_{\mathbf{p}_i}}\right)|\mathbf{p}_1,\dots \mathbf{p}_n\rangle\langle\mathbf{p}_1,\dots \mathbf{p}_n|.$$

2. *The Quantized Dirac Field* Here is some practice at manipulating anti-commutators. The quantized Dirac field and its Hamiltonian are given by

$$\begin{aligned}\psi(t,\mathbf{x}) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{s=\pm}\left(e^{-ip\cdot x}a^s_\mathbf{p}u^s(\mathbf{p}) + e^{ip\cdot x}b^{s\,\dagger}_\mathbf{p}v^s(\mathbf{p})\right)\\ H &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}E_\mathbf{p}\sum_{s=\pm}(a^{s\,\dagger}_\mathbf{p}a^s_\mathbf{p} + b^{s\,\dagger}_\mathbf{p}b^s_\mathbf{p})\end{aligned}$$

where annihilation and creation operators now satisfy anti-commutation rules. The vacuum state satisfies

$$a^s_\mathbf{p}|0\rangle = b^s_\mathbf{p}|0\rangle = 0$$

a) Check explicitly that

$$i\frac{\partial\psi}{\partial t} = [\psi, H]$$

b) Show that for a general eigenstate of the Hamiltonian, $|\psi\rangle$,

$$Ha^{s\,\dagger}_\mathbf{p}|\psi\rangle = (E_\psi + E_\mathbf{p})a^{s\,\dagger}_\mathbf{p}|\psi\rangle$$

and similarly for $b^{s\,\dagger}_\mathbf{p}|\psi\rangle$. Hence show that the spectrum of the Hamiltonian consists of states $|\{\{\mathbf{p}_i, s_i\}, i = 1\dots n\}, \{\overline{\{\mathbf{p}_i, s_i\}}, i = n+1\dots n+m\}\rangle$ with eigenvalues $\sum_{i=1}^{n+m}E_{\mathbf{p}_i}$ (here the overline denotes antiparticle states).

c) Find the three-particle wavefunction

$$\langle 0|\,\psi(x)\psi(y)\psi(z)\,|\{\mathbf{p}_1, s_1\}, \{\mathbf{p}_2, s_2\}, \{\mathbf{p}_3, s_3\}\rangle$$

Show that it can be written as a determinant (this is an example of the *Slater Determinant*) and hence is totally antisymmetric under exchange.

3. *Noether's Theorem I* The Lagrangian density for a complex scalar field is

$$\mathcal{L} = \partial^\mu\phi^\dagger\,\partial_\mu\phi - m^2\phi^\dagger\phi + \mathcal{L}_{Int}(\phi^\dagger, \phi) \tag{1}$$

a) Show that

$$\mathcal{L}_{Int} = -\frac{\lambda}{4}(\phi^\dagger\phi)^2 \tag{2}$$

does not change the conserved current arising from the global phase symmetry in the free field case.

b) Find the conserved current when

$$\mathcal{L}_{Int} = \lambda\partial^\mu(\phi^\dagger\phi)\partial_\mu(\phi^\dagger\phi) \tag{3}$$

c) What is the symmetry of the Lagrangian if

$$\mathcal{L}_{Int} = \lambda(\phi^2 + \phi^{\dagger 2})^2 \tag{4}$$

Is there a conserved current?

d) Use the phase symmetry to find the conserved current for the Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{5}$$

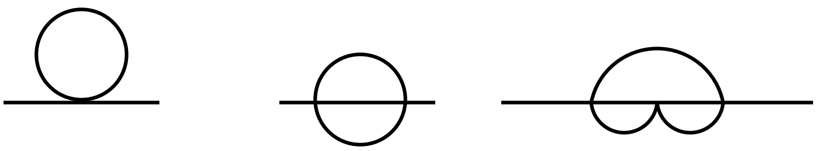
4. *Cancellation of vacuum bubble diagrams* Consider the real scalar field with interaction $\frac{1}{4!}\lambda\phi^4$ and the vacuum expectation value

$$G_K = \langle\Omega|T\prod_{i=1}^{2K}\phi(x_i)|\Omega\rangle = \frac{\langle 0|T\left(\prod_{i=1}^{2K}\phi_I(x_i)\right)\exp\left(-i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)}dtH_{Int}(\phi_I)\right)|0\rangle}{\langle 0|T\exp\left(-i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)}dtH_{Int}(\phi_I)\right)|0\rangle} \tag{6}$$

where K is an integer. Show that the contribution to G_K of $O(\lambda^L)$, L integer, in which all external points are connected to a single cluster contains no bubble diagrams.

5. *Feynman Diagrams* This is practice using the Feynman rules for scalar fields . The aim is to get the integrands for the momentum integrals with all required factors, but do not try to do the momentum integrals themselves.

a) Find expressions corresponding to the following quantum corrections to the two-point function for a single (real) scalar field.



b) Find all the distinct diagrams up to and including $O(\lambda^3)$ for the quantum corrections to the connected four-point function of the real scalar field. Be careful to label the external legs with their momenta.

c) Now consider what happens in the case of a complex scalar field with Lagrangian density

$$\mathcal{L} = \partial^\mu\phi^\dagger\,\partial_\mu\phi - m^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 \tag{7}$$

Do any of your answers in parts a) and b) change?