Lecture 7 Complex Scalar Field + Fermionic Field.

The simplest generalization of the scalar field theory is the complex scalar field. This is important because it describes charged particles. The Lagrangian density for two real scalar fields (no interactions) is

$$\mathcal{L} = \sum_{i=1,2}\left(\frac{1}{2}\,\partial^\mu\phi_i\partial_\mu\phi_i - \frac{1}{2}m^2\phi_i^2\right)$$

Now introduce

$$\left.\begin{aligned}\phi &= \frac{\phi_1 + i\phi_2}{\sqrt{2}}\\ \phi^* &= \frac{\phi_1 - i\phi_2}{\sqrt{2}}\end{aligned}\right\}$$

of course ϕ and ϕ^* both satisfy the KG equation in classical field theory

then $\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi$

The conjugate momentum is

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi^*$$

so $\mathcal{H} = \underline{\nabla}\phi^*\cdot\underline{\nabla}\phi + m^2\phi^*\phi$

We can quantize this theory in exactly the same way as before. Writing

$$\phi_i = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a^{i\,+}_{-p} + a^i_p\right)e^{ip\cdot x}$$

we get

$$\phi = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(\frac{a^1_p + ia^2_p}{\sqrt{2}} + \frac{(a^1_{-p} - ia^2_{-p})^+}{\sqrt{2}}\right)e^{ip\cdot x}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p + b^+_{-p} \right) e^{ip\cdot x}$$

$$a_p = \frac{a^1_p + i a^2_p}{\sqrt{2}}, \quad b_p = \frac{a^1_p - i a^2_p}{\sqrt{2}}$$

The Hamiltonian is of course just the sum of the Hamiltonians for ϕ_1 and ϕ_2 so

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \left(a^{1+}_p a^1_p + a^{2+}_p a^2_p \right)$$

[note that. This is set for Problem Class 1

$$a^+_p a_p + b^+_p b_p = \left(\frac{a^{1+}_p - i a^{2+}_p}{\sqrt{2}}\right)\left(\frac{a^1_p + i a^2_p}{\sqrt{2}}\right)$$
$$+ \left(\frac{a^{1+}_p + i a^{2+}_p}{\sqrt{2}}\right)\left(\frac{a^1_p - i a^2_p}{\sqrt{2}}\right)$$
$$= a^{1+}_p a^1_p + a^{2+}_p a^2_p$$
]

so $$H = \int \frac{d^3p}{(2\pi)^3} E_p \left(a^+_p a_p + b^+_p b_p \right)$$

The commutators are (using that a^1 and a^2 commute)

$$[a^+_p, a_q] = \frac{1}{2} \left[a^{1+}_p - i a^{2+}_p, a^1_q + i a^2_q \right]$$
$$= -(2\pi)^3 \delta^3(p-q)$$ as the "1" and "2" commutators add.

and similarly for b_p.

$$[b^+_p, a_q] = \frac{1}{2} \left[a^{1+}_p + i a^{2+}_p, a^1_q + i a^2_q \right]$$
$$= 0$$ this time the two non-zero quantities cancel.

From here we can go through the usual exercise of deriving time-dependent fields -

$$\phi(t, x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-iE_p t} + b^+_{-p} e^{iE_p t} \right) e^{ip\cdot x}$$

There is one ~~thing~~ feature that you should notice immediately when considering conserved quantities. As well as the total momentum $\underline{P}$ and number operator N, the combination $a_p^\dagger a_p - b_p^\dagger b_p$ will also commute with H. This reflects the existence of a <u>conserved charge</u> which is associated with the $U(1)$ invariance of $\mathcal{L}$:

$$\phi \to e^{i\alpha}\phi$$

$$\phi^* \to \phi^* e^{-i\alpha}$$

where α is independent of $(t, \underline{x})$.

<u>Noether's Theorem</u> states that associated with any continuous symmetry of a Lagrangian system there is a corresponding conserved quantity.

We can find that quantity by considering the <u>local</u> transformation $\phi \to e^{i\alpha(x)}\phi$ and asserting that the Action must be invariant.

We have

$$S = \int d^4x \left((\partial^\mu \phi^*) \partial_\mu \phi - m^2 \phi^* \phi \right)$$

$$\to \int d^4x \Big(e^{-i\alpha(x)} (\partial^\mu \phi^* - i(\partial^\mu \alpha) \phi^*)(\partial_\mu \phi + i(\partial_\mu \alpha)\phi) e^{i\alpha(x)} - m^2 \phi^* \phi \Big)$$

so $$\delta S = \int d^4x \, (\partial_\mu \alpha) \left((\partial^\mu \phi^*)\phi - \phi^* \partial^\mu \phi \right)$$

Integrating by parts and dropping boundary terms

$$\delta S = -\int d^4x\, \alpha(x)\, \partial_\mu \left((\partial^\mu \phi^*)\phi - \phi^* \partial^\mu \phi \right)$$

from which we conclude, requiring $\delta S = 0$, that $\partial_\mu j^\mu = 0$ where $j^\mu = [(\partial^\mu \phi^*)\phi - \phi^* \partial^\mu \phi] \times -i$

which of course is easily checked by computing $\partial_\mu j^\mu$ explicitly and using the equation of motion.

Now j^0 is a charge density so we can compute the total charge

$$Q = \int d^3\underline{x}\; j^0$$

$$= \int d^3\underline{x} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \cdot \frac{1}{\sqrt{2E_q}}\, e^{ip\cdot\underline{x}}\, e^{iq\cdot\underline{x}}$$

$$\times -i \Big(\left(-iE_p\, b_p\, e^{-iE_p t} + iE_p\, a^+_{-p}\, e^{iE_p t} \right) \times \left(a_q\, e^{-iE_q t} + b^+_{-q}\, e^{iE_q t} \right)$$

$$- \left(b_p\, e^{-iE_p t} + a^+_{-p}\, e^{iE_p t} \right) \times \left(-iE_q\, a_q\, e^{-iE_q t} + iE_q\, b^+_{-q}\, e^{iE_q t} \right) \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(a^+_p a_p - b^+_p b_p \right)$$

particles $\qquad$ antiparticles

Quantizing the Dirac Field

Recall that the Dirac equation can be written

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

and that the solution can be written in the form

$$\psi(t, \underline{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Big\{ e^{-i(E_p t - \underline{p}\cdot\underline{x})} \sum_{s=\pm} A_s(\underline{p}) u^s(\underline{p})$$

$$+ e^{i(E_p t + \underline{p}\cdot\underline{x})} \sum_{s=\pm} B_s(\underline{p}) v^s(\underline{p}) \Big\} \quad (*)$$

where $\pm$ denote helicity eigenspinors and $A_\pm(\underline{p})$, $B_\pm(\underline{p})$ the amplitudes for momentum $\underline{p}$

Note that $\psi(t, \underline{x})$ contains a positive energy part ($A_\pm \neq 0$) and a negative energy part ($B_\pm \neq 0$)

We showed that $u^+ u$ is not Lorentz invariant but $u^+\gamma^0 u$ is. This implies that the quantity

$$\psi^+\gamma^0 (i\gamma^\mu \partial_\mu - m)\psi$$

is Lorentz invariant and therefore a candidate for the Lagrangian Density as we expect

$$S = \int d^4x \, \mathcal{L}$$

to be Lorentz invariant, and we know d^4x is so $\mathcal{L}$ had better be too

It is conventional to define

$$\bar{\psi} = \psi^{\dagger} \gamma^{0}$$

Then we have

$$\mathcal{L} = \bar{\psi} \left(i \gamma^{\mu} \partial_{\mu} - m \right) \psi$$

from which it follows that the conjugate momentum to ψ

$$\pi = \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} = i \bar{\psi} \gamma^{0}$$

and $\mathcal{H} = \pi \, \partial_0 \psi - \mathcal{L}$

$$= \bar{\psi} \left(-i \gamma^{a} \partial_{a} + m \right) \psi \qquad a = 1, 2, 3$$

recalling that $\gamma^{\mu} = (\beta, \beta \underline{\alpha})$ this can be written

$$\mathcal{H} = \psi^{\dagger} \left(-i \underline{\alpha} . \underline{\nabla} + m \beta \right) \psi$$

– note that the quantity in brackets is exactly the original single particle q.m. Hamiltonian.

In the field theory we have

$$H = \int d^3\underline{x} \; \bar{\psi} \left(-i \gamma^{a} \partial_{a} + m \right) \psi$$

$$= \int d^3\underline{x} \; \psi^{\dagger} \left(-i \underline{\alpha} . \underline{\nabla} + m \beta \right) \psi$$

A Problem

Let's calculate H in terms of the modes.

Note that $\left(-i\gamma^a\partial_a + m\right) e^{i\underline{p}.\underline{x}}\, u^s(\underline{p})$

$$= \left(\gamma^a p_a + m\right) e^{i\underline{p}.\underline{x}}\, u^s(\underline{p})$$
$$= e^{i\underline{p}.\underline{x}} \quad E_p \gamma_0 u^s(\underline{p}).$$

and similarly $(-i\gamma^a\partial_a + m)\, e^{i\underline{p}.\underline{x}}\, v^s(\underline{p}). = -e^{i\underline{p}.\underline{x}} E_p \gamma_0 v^s(\underline{p})$

So

$$H = \int d^3\underline{x} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\underline{q}}}} e^{-i\underline{q}.\underline{x}} e^{i\underline{p}.\underline{x}}$$
$$\left(\sum_s A_s(\underline{q})^* e^{iE_{\underline{q}}t}\, \bar{u}^s(\underline{q}) + B_s(\underline{q})^* e^{-iE_{\underline{q}}t}\, \bar{v}^s(\underline{q})\right)$$
$$\times \gamma_0 E_p \left(\sum_{s'} A_{s'}(\underline{p}) e^{-iE_p t} u^{s'}(\underline{p}) \quad B_{s'}(\underline{p}) e^{iE_p t} v^{s'}(\underline{p})\right)$$
$$= \int \frac{d^3p}{(2\pi)^3} \; E_p \sum_s \left(A_s(\underline{p})^* A_s(\underline{p}) - B_s(\underline{p})^* B_s(\underline{p})\right)$$

where we have used the normalization $u^+ u = +2E_p$, $v^+ v = +2E_p$

(see Lecture 2).

You can see straightaway that we have a Big Problem. This Hamiltonian does not look +ve definite. If we replace $A_s(\underline{p}) \to a^s_{\underline{p}}$, $A_s(\underline{p})^* \to a^{s+}_{\underline{p}}$ etc in the usual way, & with a's and b's satisfying commutation rules there will be no ground state. Most textbooks contain a discussion of further consequences, there is a good one in Peskin + Schroder.

The Solution

We have to abandon the notion that annihilation and creation operators satisfy commutation rules. Fermionic fields satisfy anti-commutation rules. We write

$$\psi(t,\underline{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\underline{p}\cdot\underline{x}}}{\sqrt{2E_p}} \sum_s e^{-iE_pt} a_{\underline{p}}^s u^s(\underline{p}) + e^{iE_pt} b_{-\underline{p}}^{s+} v^s(-\underline{p})$$

By making the change of variables $\underline{p} \to -\underline{p}$ in the second term we can rewrite ψ in the form

$$\psi(t,\underline{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s e^{-ip\cdot x} a_{\underline{p}}^s u^s(\underline{p}) + e^{ip\cdot x} b_{\underline{p}}^{s+} v^s(\underline{p})$$

where $p = (E_p, \underline{p})$. It follows that

$$\bar{\psi}(t,\underline{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s e^{-ip\cdot x} \bar{v}^s(\underline{p})\, b_{\underline{p}}^s + e^{ip\cdot x} \bar{u}^s(\underline{p})\, a_{\underline{p}}^{s+}$$

The creation and annihilation operators satisfy anticommutation $\{Q_1, Q_2\} = Q_1 Q_2 + Q_2 Q_1$

$$\{a_{\underline{p}}^{r}, a_{\underline{q}}^{s+}\} = \{b_{\underline{p}}^{r}, b_{\underline{q}}^{s+}\} = (2\pi)^3 \, \delta^3(\underline{p}-\underline{q}) \, \delta^{rs}$$

all other combinations anticommute

We will explore the implications next time.