Lecture 8 Properties of the Dirac Quantum Field

The Hamiltonian is

$$H = \int d^3\underline{x} \;\; \bar{\psi}(x)(-i\gamma^a\partial_a + m)\psi(x) \quad , \quad a=1,2,3$$

with

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s e^{-ip\cdot x} a^s_p \, u^s(p) + e^{ip\cdot x} b^{s+}_p \, v^s(p)$$

so

$$(-i\gamma^k\partial_k + m)\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (\gamma^a p_a + m) u^s(p)\, a^s_p e^{-ip\cdot x} + (-\gamma^a p_a + m) v^s(p)\, b^{s+}_p e^{ip\cdot x}$$

$k=1,2,3$

recall that $(\gamma^k p_k + m) u^s(p) = E_p \gamma^0 u^s(p)$

and $(\gamma^k p_k + m) v^s(-p) = -E_p \gamma^0 v^s(-p)$

so $(-\gamma^k p_k + m) v^s(p) = -E_p \gamma^0 v^s(p)$.

giving us

$$(-i\gamma^k\partial_k + m)\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{E_p\gamma^0}{\sqrt{2E_p}} \sum_s e^{-ip\cdot x} a^s_p \, u^s(p) - e^{ip\cdot x} b^{s+}_p \, v^s(p)$$

$$\bar{\psi}(x) = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \sum_{s'} e^{-ip'\cdot x} b^{s'}_{p'} \, \bar{v}^{s'}(p') + e^{ip'\cdot x} a^{s'+}_{p'} \, \bar{u}^{s'}(p')$$

when we combine these to get H

$$H = \int d^3\underline{x}\, \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \; E_p \sum_{ss'}$$

$$\left(e^{-ip'\cdot x}\, b^{s'}_{p'}\, v^{+s'}(p') + e^{ip'\cdot x} a^{s'+}_{p'}\, u^{+s'}(p') \right)$$

$$\times \left(e^{-ip\cdot x}\, a^{s}_{p}\, u^{s}(p) - e^{ip\cdot x}\, b^{s+}_{p}\, v^{s}(p) \right).$$

$\int d^3\underline{x}$ will give us $(2\pi)^3\, \delta^3(p \pm p')$ depending on the exponential so we can then always do $\int \frac{d^3p'}{(2\pi)^3}$ noting that $E_{-p} = E_p$ to get

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \; E_p \sum_{ss'}$$

$$\times \Big[a^{s'+}_{p}\, a^{s}_{p}\, u^{+s'}(p)\, u^{s}(p) - b^{s'}_{p}\, b^{s+}_{p}\, v^{+s'}(p) v^{s}(p)$$
$$+ b^{s'}_{-p}\, a^{s}_{p}\, e^{-i2E_p t}\, v^{+s'}(-p)\, u^{s}(p)$$
$$- a^{s'+}_{-p}\, b^{s+}_{p}\, e^{i2E_p t}\, u^{+s'}(-p)\, v^{s}(p) \Big]$$

now $\quad u^{+s'}(p)\, u^{s}(p) = 2E_p\, \delta^{ss'} = v^{+s'}(p)\, v^{s}(p)$

and $\quad v^{+s'}(-p)\, u^{s}(p) = v^{+\hat{s}'}(p)\, u^{s}(p) = 0$ etc.

we get $\quad H = \displaystyle\int \frac{d^3p}{(2\pi)^3}\, E_p \sum_{s} a^{s+}_{p}\, a^{s}_{p} - b^{s}_{p}\, b^{s+}_{p}$

using the anticommutator, and dropping the infinite vacuum energy

$$H = \int \frac{d^3p}{(2\pi)^3} \, E_p \sum_s a^{s\dagger}_p a^s_p + b^{s\dagger}_p b^s_p$$

The vacuum state in annihilated by

$$a^s_p |0\rangle = b^s_p |0\rangle = 0$$

and are particle states, which now have a spin label as well as p

$$|p, s\rangle = \sqrt{2E_p} \, a^{\dagger s}_p |0\rangle$$

and

$$|\overline{p, s}\rangle = \sqrt{2E_p} \, b^{\dagger s}_p |0\rangle$$

are <u>are anti-particle</u> states.

It is left as an exercise to show that

$$\underline{P} = \int d^3\underline{x} \; \psi^\dagger (-i \underline{\nabla}) \psi = \int \frac{d^3p}{(2\pi)^3} \, p \sum_s a^{s\dagger}_p a^s_p + b^{s\dagger}_p b^s_p$$

and that the conserved charge density

$$Q = \int d^3x \; \psi^\dagger \psi = \int \frac{d^3p}{(2\pi)^3} \sum_s a^{s\dagger}_p a^s_p - b^{s\dagger}_p b^s_p$$

It is important to be clear that, looking back to our definition of ψ on p7.8

a^s_p removes a particle of spin s, momentum p, energy E_p

b^{s+}_p adds an anti-particle of spin s, momentum p, energy E_p

so removing a particle or adding an antiparticle

Causality

We should still have the principle that only events that are not space-like separated can be in causal contact. Space like separated fermionic operators should anti-commute. So we calculate

$$\{ \bar{\psi}_a(x),\ \psi_b(y) \}$$

where a and b now denote the spinor components.
so $a, b = 1, 2, 3, 4$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \sum_{ss'}$$

$$\Big\{ e^{-ip'\cdot x}\, b^{s'}_{p'}\, \bar{v}^{s'}_a(p') + e^{ip'\cdot x}\, a^{s'+}_{p'}\, \bar{u}^{s'}_a(p'),$$

$$e^{-ip\cdot y}\, a^s_p\, u^s_b(p) + e^{ip\cdot y}\, b^{s+}_p\, v^s_b(p) \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \sum_{ss'}$$

$$\Big(e^{i(p\cdot y - p'\cdot x)} (2\pi)^3 \delta^3(p-p') \delta^{ss'} \bar{v}_a^{s'}(p') v_b^s(p)$$
$$+ e^{-i(p\cdot y - p'\cdot x)} (2\pi)^3 \delta^3(p-p) \delta^{ss'} \bar{u}_a^{s'}(p') u_b^s(p) \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \begin{pmatrix} e^{ip\cdot(y-x)} \bar{v}_a^s(p) v_b^s(p) \\ + e^{-ip\cdot(y-x)} \bar{u}_a^s(p) u_b^s(p) \end{pmatrix}$$

We have now to evaluate the spin sums

$$\sum_s u_b^s(p) \bar{u}_a^s(p)$$

$$= \sum_{s=\pm} \begin{pmatrix} \sqrt{E_p - s|p|}\ \xi^s(p) \\ \sqrt{E_p + s|p|}\ \xi^s(p) \end{pmatrix} \cdot \left(\sqrt{E_p + s|p|}\ \xi^s(p)^\dagger, \sqrt{E_p - s|p|}\ \xi^s(p)^\dagger \right)$$

$$= \sum_{\substack{s=\pm \\ a', b' = 1,2}} \begin{pmatrix} m\ \xi^s_{b'}(p) \xi^s_{a'}(p)^\dagger & (E_p - s|p|) \xi^s_{b'}(p) \xi^s_{a'}(p)^\dagger \\ (E_p + s|p|) \xi^s_{b'}(p) \xi^s_{a'}(p)^\dagger & m\ \xi^s_{b'}(p) \xi^s_{a'}(p)^\dagger \end{pmatrix}$$

$$\cdot \sum_{s=\pm} \xi^s_{b'}(p) \xi^s_{a'}(p)^\dagger = \xi^+_{b'}(p) \xi^+_{a'}(p)^\dagger + \xi^-_{b'}(p) \xi^-_{a'}(p)^\dagger$$

$$= \delta_{b'a'}$$

because $\xi^\pm(p)$ span a two-dimensional complex vector space and are normalized to one (ie $|-\rangle\langle -| + |+\rangle\langle +| = \mathbb{1}$)

then note that

$$(\underline{\sigma}\cdot \underline{p})_{a'c'}\, \xi^{\pm}_{c'}(p) = \pm |p|\, \xi^{\pm}_{a'}(p)$$

$$\sum_{s=\pm} \xi^{s\dagger}_{b'}(p)(\underline{\sigma}\cdot \underline{p})_{a'c'}\, \xi^{s}_{c'}(p) = \sum_{s=\pm} \pm s|p|\, \xi^{s}_{a'}(p)\, \xi^{s}_{b'}(p)^{\dagger}$$

$$(\underline{\sigma}\cdot \underline{p})_{a'c'}\, \delta_{b'c'} = (\underline{\sigma}\cdot \underline{p})_{a'b'}$$

So we have

$$\sum_s u^s_b(p)\, \bar{u}^s_a(p) = \begin{pmatrix} m & E_p - \underline{\sigma}\cdot\underline{p} \\ E_p + \underline{\sigma}\cdot\underline{p} & m \end{pmatrix}$$

$$= (m + \not{p})_{ba}$$

Where we have introduced the notation $\not{p} = \gamma^\mu p_\mu$

Now note that because $u^\pm, v^\pm$ span a 4-dim complex vector space and are normalized to $2E_p$

$$\sum_s u^s_b(p)\, u^{s\dagger}_c(p) + v^s_b(-p)\, v^{s\dagger}_c(-p) = 2E_p\, \delta_{bc}.$$

so $$\sum_s v^s_b(-p)\, \bar{v}^s_c(-p) = 2E_p\, \gamma^0_{bc} - (\not{p}+m)_{ba}$$

$$\sum_s v^s_b(p)\, \bar{v}^s_c(p) = (\not{p}-m)_{ba}$$

Putting these results (which are of far more general utility as we will see in due course) into the anticommutator gives

$$\{\bar{\psi}_a(x), \psi_b(y)\} = (i\not{\partial}_y + m)_{ba} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}\left(e^{-ip\cdot(y-x)} - e^{ip\cdot(y-x)}\right).$$

$$= (i\not{\partial}_y + m)_{ba}\, i\, \Delta(y-x)$$

so has the same causality properties as $\Delta(y-x)$! All is well. ~~##~~ As in the scalar case the (anti-) commutator is closely related to the retarded Green's function of the Dirac operator

We first note that

$$S_R(y-x) = (i\not{\partial}_y + m)\, D_R(y-x).$$

retarded Green's function for scalar field.

is a Green's function of the Dirac operator because

$$(i\not{\partial}_y - m)\, S_R(y-x) = -(\Box_y + m^2)\, D_R(y-x)$$

$$= i\,\delta^4(y-x)$$

Now $\quad D_R(y-x) = \Theta(y^0-x^0)\, i\,\Delta(y-x)$

so $\quad S_R(y-x) = (i\not{\partial}_y + m)\; i\,\Theta(y^0-x^0)\,\Delta(y-x)$

$$= -\,\delta(y^0-x^0)\,\Delta(y-x)$$

$$+\, i\,\Theta(y^0-x^0)\,(i\not{\partial}_y + m)\,\Delta(y-x)$$

The first term vanishes because when $y^0 = x^0$

$$\Delta(y-x) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\left(e^{i\underline{p}\cdot(\underline{y}-\underline{x})} - e^{-i\underline{p}\cdot(\underline{y}-\underline{x})}\right)$$

set $\underline{p} \to -\underline{p}$ in this term

$$= 0$$

Hence $\quad S_R(y-x)_{ba} = \Theta(y^0-x^0)\,\{\psi_b(y), \bar{\psi}_a(x)\}$

so it is indeed the retarded Green's function.

Finally $\quad S_R(y-x) = (i\not{\partial}_y + m)\,\underbrace{\int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2}\,e^{-ip\cdot(y-x)}}_{\text{4 dim. integral for } D_R}$

$$= \int \frac{d^4p}{(2\pi)^4}\; i\,\frac{\not{p}+m}{p^2-m^2}\,e^{-ip\cdot y}$$

with the same pole prescription as for D_R