Lecture 9 Interacting Field Theories

So far we have just described free particles using QFT. This is fine but ultimately not very interesting. To understand how to incorporate interactions between these particles go back to the scalar field theory and consider the potential

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4$$

so in the original lattice picture we have an anharmonic oscillator on each site. Then the Hamiltonian operator will be modified to

$$H = H_0 + H_{int}$$

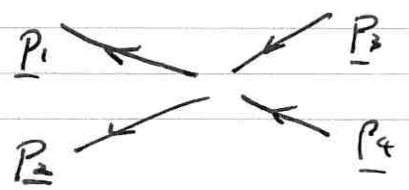
$$H_{int} = \frac{\lambda}{4!} \int d^3\underline{x} \prod_{i=1}^{4} \int \frac{d^3\underline{p}_i}{(2\pi)^3} \frac{1}{\sqrt{2E_{p_i}}} \left(a_{\underline{p}_i} e^{-ip_i \cdot x} + a^\dagger_{\underline{p}_i} e^{ip_i \cdot x} \right)$$

Focus on the terms containing 2 $a^\dagger$s and 2 as. They take the form

$$\frac{\lambda}{4!} \prod_i \int \frac{d^3\underline{p}_i}{(2\pi)^3} \frac{1}{\sqrt{2E_{p_i}}} \int d^3\underline{x} \; a^\dagger_{\underline{p}_1} a^\dagger_{\underline{p}_2} a_{\underline{p}_3} a_{\underline{p}_4} \times e^{i(p_1+p_2-p_3-p_4)\cdot x}$$

$$= \frac{\lambda}{4!} \prod_i \int \frac{d^3\underline{p}_i}{(2\pi)^3} \frac{1}{\sqrt{2E_{p_i}}} (2\pi)^3 \, \delta^3(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4) \times e^{it(E_1+E_2-E_3-E_4)} a^\dagger_{\underline{p}_1} a^\dagger_{\underline{p}_2} a_{\underline{p}_3} a_{\underline{p}_4}$$

This term annihilates two particles and creates two new ones ~~with a~~ while conserving momentum



when we do the full calculation properly we will find that the energy is also conserved in such a two particle scattering process.

Interaction Picture

We start with $H = H_0 + H_{int}$

expressed in terms of the full ~~interacting~~ field $\phi(x)$ and with exact vacuum $|\Omega\rangle$ so

$$H|\Omega\rangle = E_\Omega |\Omega\rangle$$

We aim to calculate

$$\langle\Omega| T \prod_i \phi(x_i) |\Omega\rangle \qquad (*)$$

where the time ordering operator T is an instruction to write the fields ϕ in order where highest times go to the left. So if

$\phi(x_1)\phi(x_2)\phi(x_3)$ is T ordered $x_1^0 > x_2^0 > x_3^0$

The interaction representation field $\Phi_I(t, \underline{x})$ is a free field which evolves according to

$$\Phi_I(t, \underline{x}) = e^{iH_0(t-t')} \Phi_I(t', \underline{x}) e^{-iH_0(t-t')}$$

Our aim is to express $(*)$ in terms of ϕ_I.

Now set the full field at t_0

$$\phi(t_0, \underline{x}) = \phi_I(t_0, \underline{x})$$

then $$\phi(t, \underline{x}) = e^{iH(t-t_0)} \phi_I(t_0, \underline{x}) e^{-iH(t-t_0)}$$

$$= U^{+}(t,t_0)\ \phi_I(t_0, x)\ U(t,t_0)$$

where

$$U(t,t_0) = e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}$$

and $U(t,t_0)$ is unitary — $U(t,t_0)(U(t,t_0))^{+} = 1$

Note that H and H_0 don't commute so we have to be careful in computing U. Note

$$\frac{\partial U}{\partial t} = -i\, e^{iH_0(t-t_0)} (H - H_0)\, e^{-iH(t-t_0)}$$

$$= -i\, \underbrace{e^{iH_0(t-t_0)} (H - H_0)\, e^{-iH_0(t-t_0)}}_{\text{evolving with } H_0} \times e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}$$

$$= -i\ H_{int}(\phi_I)\ U(t,t_0)$$

because at $t=t_0$, $H - H_0 = H_{int}(\phi_I(t_0))$ (remember $\phi = \phi_I$ at $t=t_0$), and this is then evolved with H_0 from t_0 to t which is exactly the way ϕ_I evolves so we end up with $H_{int}(\phi_I(t))$

It is easy to see (by differentiating) that the solution is

$$
\begin{aligned}
U = 1 - i &\int_{t_0}^{t} dt_1\, H_{int}(\phi_I(t_1)) \\
&+ (-i)^2 \int_{t_0}^{t} dt_1\, H_{int}(\phi_I(t_1)) \int_{t_0}^{t_1} dt_2\, H_{int}(\phi_I(t_2)) \\
&+ (-i)^3 \int_{t_0}^{t} dt_1\, H_{int}(\phi_I(t_1)) \int_{t_0}^{t_1} dt_2\, H_{int}(\phi_I(t_2)) \int_{t_0}^{t_2} dt_3\, H_{int}(\phi_I(t_3)) \\
&+ \ldots \qquad (*)
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\partial U}{\partial t} &= -i\, H_{int}(\phi_I(t)) \\
&\quad + (-i)^2 H_{int}(\phi_I(t)) \int_{t_0}^{t} dt_2\, H_{int}(\phi_I(t_2)) \\
&\quad + (-i)^3 H_{int}(\phi_I(t)) \int_{t_0}^{t} dt_2\, H_{int}(\phi_I(t_2)) \int_{t_0}^{t_2} dt_3\, H_{int}(\phi_I(t_3)) \\
&\quad + \ldots \\
&= -i\, H_{int}(\phi_I(t))\, U
\end{aligned}
$$

as claimed. Now note that in $(*)$ the times are ordered so that $t > t_1 > t_2 > \ldots$; therefore it is actually a time ordered product so in fact

$$
U = T\left(1 - i\int_{t_0}^{t} dt_1\, H_{int}(\phi_I(t_1)) + \ldots\right) = T(*)
$$

The region $t > t_1 > t_2 > \ldots > t_n > 0$

spans a portion of the hypercube $[0,t]^n$ which

is $f_n = \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$

$$= \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-2}} dt_{n-1} \, (t_{n-1}).$$

$$= \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-3}} dt_{n-2} \, \frac{1}{2} t_{n-1}^2$$

$$\vdots$$

$$= \frac{t^n}{n!}$$

So in our expression for U we can replace all the upper integration limits provided we divide the $O(n)$ (in H_{int}) term by $n!$ The T product ensures that everything still comes out in the correct time ordering so we have shown that

$$U(t,t_0) = T \exp\left(-i \int_{t_0}^{t} H_{int}(\phi_I(t'))\, dt'\right)$$

It follows that $\quad U(t,t')\, U(t',t'') = U(t,t'') \qquad (1)$

then also $\quad U(t,t') \underbrace{U(t',t'')\, U^{+}(t',t'')}_{1} = U(t,t'')\, U^{+}(t',t'')$

$$U(t,t') = U(t,t'')\, U^{+}(t',t'') \qquad (2)$$

and $\quad U^{+}(t,t') = U(t',t'')\, U^{+}(t,t'') \qquad (3)$

We now know how to relate the full field $\phi(x)$ to the interaction picture field $\phi_I(x)$ which is a free field. Next we need to find the true vacuum $|\Omega\rangle$. Consider

$$e^{-iH\tilde{t}}|0\rangle = \sum_n e^{-iE_n\tilde{t}}|n\rangle\langle n|0\rangle$$

where the states $|n\rangle$ are eigenstates of H, $|\Omega\rangle$ corresponding to $n=0$. We can separate out the first term to get, <u>assuming $|\Omega\rangle$ is unique</u>,

$$e^{-iH\tilde{t}}|0\rangle = e^{-iE_\Omega\tilde{t}}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n>0} e^{-iE_n\tilde{t}}|n\rangle\langle n|0\rangle$$

Now set $\tilde{t} = T(1-i\epsilon) + t_0$ $\quad\epsilon$ small >0

and define

$$|\Omega_T\rangle = \frac{e^{-iH\tilde{t}}|0\rangle}{e^{-iE_\Omega\tilde{t}}\langle\Omega|0\rangle}$$

$$= \frac{|\Omega\rangle\langle\Omega|0\rangle + \sum_{n>0} e^{-i(E_n-E_\Omega)(T(1-i\epsilon)+t_0)} \times |n\rangle\langle n|0\rangle}{\langle\Omega|0\rangle}$$

If we take $T \to +\infty$ then ~~the~~ the $n \neq 0$ terms contain a factor $e^{-\epsilon(E_n - E_\Omega)T} \to 0$ because $E_n - E_\Omega > 0$. Hence

$$|\Omega_\infty\rangle = |\Omega\rangle$$

Now, because $H_0|0\rangle = 0$,

$$|\Omega_T\rangle = \frac{e^{-iH\tilde{t}}\, e^{iH_0\tilde{t}}\,|0\rangle}{e^{-iE_\Omega \tilde{t}}\langle\Omega|0\rangle}$$

$$= \frac{e^{-iH(t_0 - (-T(1-i\epsilon)))} \times e^{iH_0(t_0 - (-T(1-i\epsilon)))}\,|0\rangle}{e^{-iE_\Omega\tilde{t}}\langle\Omega|0\rangle}$$

but $U(t,t_0) = e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}$

so $U^+(t,t_0) = e^{-iH(t_0-t)}\, e^{iH_0(t_0-t)}$

and $|\Omega_T\rangle = \dfrac{U^+(-T(1-i\epsilon), t_0)\,|0\rangle}{e^{-iE_\Omega\tilde{t}}\langle\Omega|0\rangle}$

It is sufficient to leave $|\Omega_T\rangle$ in this form.

A similar exercise shows that defining

$$\langle\Omega_T| = \frac{\langle 0|\, U(T(1-i\epsilon), t_0)}{e^{iE_\Omega(t_0 - T(1-i\epsilon))}\langle 0|\Omega\rangle}$$

we have $\langle\Omega_\infty| = \langle\Omega|$

Now we can assemble everything. First

$$\langle \Omega_T | \Omega_T \rangle = \frac{\langle 0 | U(T(1-i\epsilon), t_0)\; U^\dagger(-T(1-i\epsilon), t_0) | 0 \rangle}{e^{-2iE_\Omega T(1-i\epsilon)}\; |\langle 0 | \Omega \rangle|^2}$$

Using (2) on page 9.5 we get

$$\langle \Omega_T | \Omega_T \rangle = \frac{\langle 0 | U(T(1-i\epsilon), -T(1-i\epsilon)) | 0 \rangle}{e^{-2iE_\Omega T(1-i\epsilon)}\; |\langle 0 | \Omega \rangle|^2}$$

Then for the time ordered product of two fields

$$\langle \Omega_T | T\, \phi(x)\, \phi(y) | \Omega_T \rangle$$

$$= \langle \Omega_T | \phi(x)\, \phi(y) | \Omega_T \rangle \quad \text{assuming } x^0 > y^0 > t^0$$

$$= \frac{\langle 0 | U(T(1-i\epsilon), t_0)\, U^\dagger(x^0, t_0)\, \phi_I(x^0)\, U(x^0, t_0) \times U^\dagger(y^0, t_0)\, \phi_I(y^0)\, U(y^0, t_0)\, U^\dagger(-T(1-i\epsilon), t_0) | 0 \rangle}{e^{-2iE_\Omega T(1-i\epsilon)}\; |\langle 0 | \Omega \rangle|^2}$$

Again using (2) this gives

$$= \frac{\langle 0 | U(T(1-i\epsilon), x^0)\, \phi_I(x^0)\, U(x^0, y^0)\, \phi_I(y^0)\, U(y_0, -T(1-i\epsilon) | 0 \rangle}{e^{-2iE_\Omega T(1-i\epsilon)}\; |\langle 0 | \Omega \rangle|^2}$$

so finally, noting that if $y^0 > x^0$ it just comes out with $\cancel{\phi_I(x^0) \Leftrightarrow}$ $x^0 \Leftrightarrow y^0$ we get

$$\frac{\langle \Omega_T | T\, \phi(x)\, \phi(y) | \Omega_T \rangle}{\langle \Omega_T | \Omega_T \rangle} = \frac{\langle 0 | T\, \phi_I(x)\, \phi_I(y)\, e^{-i\int_{-T(1-i\epsilon)}^{T(1-i\epsilon)} H_{int}(\phi_I(t))\, dt} | 0 \rangle}{\langle 0 | T\, e^{-i\int_{-T(1-i\epsilon)}^{T(1-i\epsilon)} H_{int}(\phi_I(t))\, dt} | 0 \rangle}$$

Taking $T \to \infty$ we get

$$\frac{\langle \Omega | T\, \phi(x)\, \phi(y) | \Omega \rangle}{1} = \lim_{T \to \infty} \left(\quad \downarrow \quad \right).$$

It is easy to see that this extends to any time ordered product. We can use this expression to generate a perturbation expansion in powers of H_{int} for time ordered products.

Two Point Function

We will start with the two point function. The methods that we develop will then extend immediately to higher order correlation functions. The lowest order approximation to $\langle\Omega| T\,\phi(x)\,\phi(x')|\Omega\rangle$ is just the free field object

$$D_F(x-x') = \langle 0|\, T\, \phi_I(x)\, \phi_I(x')\, |0\rangle$$

This is the Feynman propagator and is of fundamental importance in everything that follows.

We need to compute

$$D_F(x-x') = \langle 0| T(\phi(x)\ \phi(x'))|0\rangle$$
$$= \theta(t'-t)\ \langle 0|\ \phi(x')\phi(x)|0\rangle + \theta(t-t')\langle 0|\phi(x)\ \phi(x')|0\rangle$$

Now we can compute

$$\langle 0|\ \phi(x')\ \phi(x)|0\rangle = \langle 0| \int \frac{d^3\underline{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\underline{p}}}}\ \frac{d^3\underline{q}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\underline{q}}}}\ \cdot$$

$$\left(a^+_{\underline{p}}\ e^{ip\cdot x'} + a_{\underline{p}}\, e^{-ip\cdot x'}\right)\left(a^+_{\underline{q}}\ e^{iq\cdot x} + a_{\underline{q}}\, e^{-iq\cdot x}\right)|0\rangle$$

$$= \int \frac{d^3\underline{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\underline{p}}}}\ \frac{d^3\underline{q}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\underline{q}}}}\ e^{i(q\cdot x - p\cdot x')}\ \langle 0|\ a_{\underline{p}}\, a_{\underline{q}}^+|0\rangle$$

but $\langle 0|\, a_{\underline{p}} a_{\underline{q}}^+|0\rangle = \langle 0|\,[a_{\underline{p}}, a_{\underline{q}}^+]\,|0\rangle = (2\pi)^3\ \ \delta^3(\underline{p}-\underline{q})$ so

$$\langle 0|\phi(x')\ \phi(x)|0\rangle = \int\frac{d^3\underline{p}}{(2\pi)^3}\ \frac{1}{2E_{\underline{p}}}\ e^{ip\cdot(x-x')}$$

and so

$$D_F(x-x') = \int \frac{d^3\underline{p}}{(2\pi)^3}\ \frac{1}{2E_p}\left\{\ \theta(t'-t)\ e^{ip\cdot(x-x')} + \theta(t-t')\, e^{-ip\cdot(x-x')}\right\}$$

Remember that here ~~$p_0 = E_p$~~ $p_0 = E_{\underline{p}}$

It is very easy to see that an alternative ~~metho~~ way of writing D_F is

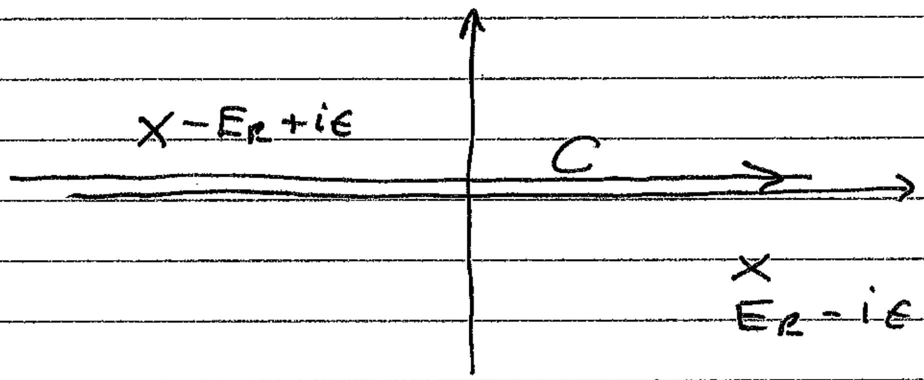
$$D_F(x-x') = \int \frac{d^4p}{(2\pi)^4}\ \frac{i\ e^{-ip\cdot(x-x')}}{p^2-m^2+i\epsilon}$$

where here p_0 is just an integration variable

~~Next problem sheet~~

$$= \int \frac{d^3p}{(2\pi)^3} e^{i p\cdot(\underline{x}-\underline{x}')} \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dp_0 \, e^{-i p_0 (t-t')}}{p_0^2 - E_p^2 + i\epsilon}$$

There are poles at $p_0 = \pm\sqrt{E_p^2 - i\epsilon}$



We close the integration contour C

i) in the upper half plane if $t-t' < 0$

ii) in the lower half plane if $t-t' > 0$

Using the residue theorem this gives

$$D_F(x-x') = \int \frac{d^3p}{(2\pi)^3} e^{i p\cdot(\underline{x}-\underline{x}')} \frac{i}{2\pi} \cdot 2\pi i$$

$$\times \left\{ \theta(t'-t) \frac{1}{-2E_p} e^{iE_p(t-t')} + \underset{\substack{\uparrow \\ \text{clockwise contour has } (-1) \text{ factor}}}{(-1)} \theta(t-t') \frac{1}{2E_p} e^{-iE_p(t-t')} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left\{ \theta(t'-t) e^{ip\cdot(x-x')} + \theta(t-t') e^{-ip(x} \right\}$$

where now $p^0 = E_p$ and we have made a change of variables $p \to -p$ in the first term