Lecture 10 The Perturbation Expansion

We can use the formalism we developed last time to generate a perturbation expansion. For definiteness consider $H_{int} = \int d^3x \frac{\lambda}{4!} \phi_I^4$ then

$$\langle \Omega | T\, \phi(x)\, \phi(y) | \Omega \rangle = \frac{\langle 0 | T\, \phi_I(x)\, \phi_I(y)\, e^{-i \int d^4x' \frac{\lambda}{4!} \phi_I(x')^4} | 0 \rangle}{\langle 0 | T\, e^{-i \int d^4x' \frac{\lambda}{4!} \phi_I(x')^4} | 0 \rangle}$$

Remember that $\phi_I$ is a free-field. We will drop the $I$ subscript from now on. If we expand the exponential in powers of $\lambda$ we get

$$\langle \Omega | T\, \phi(x)\, \phi(y) | \Omega \rangle = \frac{\langle 0 | T\, \phi(x)\, \phi(y) \left(1 - i \frac{\lambda}{4!} \int d^4x'\, \phi(x')^4 + \dots \right) | 0 \rangle}{\langle 0 | T\; 1 - i \frac{\lambda}{4!} \int d^4x'\, \phi(x')^4 + \dots \; | 0 \rangle} \quad (*)$$

Now since $\phi$ is a free field this expression is definitely computable but I hope you can see that it will soon get very messy. We need a way of organizing it. This is provided by Wick's Theorem.

let $A(x)$ stand for some general bosonic field

then $A(x) = A^+(x) + A^-(x)$

$\nearrow$ creation part $\uparrow$ annihilation part

and $A(x_1)\,A(x_2) = \left(A^+(x_1) + A^-(x_1)\right)\left(A^+(x_2) + A^-(x_2)\right)$

$$= A^+(x_1)A^+(x_2) + A^+(x_1)A^-(x_2)$$
$$+ A^-(x_1)A^+(x_2) + A^-(x_1)A^-(x_2)$$

now $:A(x_1)\,A(x_2): = A^+(x_1)A^+(x_2) + A^+(x_1)A^-(x_2)$
$$+ A^+(x_2)A^-(x_1) + A^-(x_1)A^-(x_2)$$

so $A(x_1)A(x_2) = :A(x_1)A(x_2): + [A^-(x_1), A^+(x_2)]$

but the commutator is a c-number so equal to its vacuum expectation value

$$A(x_1)A(x_2) = :A(x_1)A(x_2): + \langle 0|\, A^-(x_1)A^+(x_2) - \cancel{A^+(x_2)A^-(x_1)|0\rangle}$$
$$\swarrow 0$$

$$= :A(x_1)A(x_2): + \langle 0|\, A(x_1)A(x_2)\,|0\rangle$$

Now consider

$$T(A(x_1)\,A(x_2))$$

$$= \theta(t_1 - t_2)\,A(x_1)\,A(x_2) + \theta(t_2 - t_1)\,A(x_2)\,A(x_1)$$

now write

$$A(x_1)\,A(x_2) = \,:A(x_1)\,A(x_2): + \langle 0|\,A(x_1)\,A(x_2)\,|0\rangle$$

~~which defines normal ordering~~

then

$$T(\ldots\ldots) = \,:A(x_1)\,A(x_2):\,\big(\theta(t_1 - t_2) + \theta(t_2 - t_1)\big)$$

$$+ \langle 0|\,\theta(t_1 - t_2)\,A(x_1)\,A(x_2) + \theta(t_2 - t_1)\,A(x_2)\,A(x_1)\,|0\rangle$$

$$= \,:A(x_1)\,A(x_2): + \langle 0|\,T(A(x_1)\,A(x_2))\,|0\rangle$$

which is our first result.

You might like to <u>repeat the exercise for yourselves</u> to show that

$$T\left(A(x_1)\,A(x_2)\,A(x_3)\right)$$

$$= :A(x_1)\,A(x_2)\,A(x_3): \;+\; :A(x_1):\,\langle 0|\,T\left(A(x_2)\,A(x_3)\right)|0\rangle$$

$$+ :A(x_2):\,\langle 0|\,T\left(A(x_1)\,A(x_3)\right)|0\rangle$$

$$+ :A(x_3):\,\langle 0|\,T\left(A(x_1)\,A(x_2)\right)|0\rangle$$

The general result is

$$T\left(A(x_1)\ldots A(x_n)\right) = \;:A(x_1)\ldots\ldots A(x_n):$$

$$+\sum_{\substack{\text{pairs}\\ ij}} :A(x_1)\ldots\widetilde{A(x_i)}\ldots\widetilde{A(x_j)}\ldots A(x_n):\,\langle 0|\,T\left(A(x_i)\,A(x_j)\right)|0\rangle$$

$$+\sum_{\substack{\text{two pairs}\\ i<j,\,k<l}} :\text{two pairs missing}:\,\langle 0|\,T\left(A(x_i)\,A(x_j)\right)|0\rangle$$

$$\times\,\langle 0|\,T\left(A(x_k)\,A(x_l)\right)|0\rangle$$

$$+\;\ldots\ldots\ldots\; + \sum_{\text{all pairings}}\prod_{lk}\langle 0|TA(x_l)\,A(x_k)\,|0\rangle\ldots$$

This may be proved by <u>induction</u> : we will not do that here because it takes all day on a blackboard.

Now back to (*). First examine the denominator.

It is $1 - \frac{i\lambda}{4!}\int \langle 0|T\ \phi(x')\ \phi(x')\ \phi(x')\ \phi(x')\ |0\rangle\, d^4x'$

we contract the fields by pairs. There are exactly three ways of doing it so we get

$$1 - \frac{i\lambda.3}{4!}\int d^4x' \left(\langle 0|T\phi(x')\,\phi(x')|0\rangle\right)^2 + \dots$$

$$= 1 - i\,\frac{3\lambda}{4!}\int d^4x'\, D_F(x'-x')^2 + \dots$$

This looks a bit odd but let's carry on with the numerator. It is

$$\langle 0|T\,\phi(x)\,\phi(y)|0\rangle - \frac{i\lambda}{4!}\int \langle 0|T\phi(x)\,\phi(y)\ \phi(x')^4|0\rangle d^4x'$$

are contracting pairs $\phi(x)$, and $\phi(y)$ the the 4 of $\phi(x')$ fields among themselves – this is the same calculation as above. So we get

$$\langle 0|T\phi(x)\,\phi(y)|0\rangle \times \underbrace{\left(1 - i\,\frac{3\lambda}{4!}\int d^4x'\, D_F(0)^2\right)}_{\text{same}}$$

$$-\frac{i\lambda}{4!}\int 4\,\langle 0|T\,\phi(x)\ \phi(x')|0\rangle \times 3\,\langle 0|T\phi(y)\,\phi(x')|0\rangle$$
$$\times \langle 0|T\,\phi(x')\,\phi(x')|0\rangle\ d^4x'$$

Note that the term in brackets cancels with the denominator and formally to $O(\lambda)$ we are left with

$$D_F(x-y) \; -\frac{i\lambda}{2} \; D_F(0) \int D_F(x-x')\, D_F(y-x')\, d^4x'$$

But $D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$.

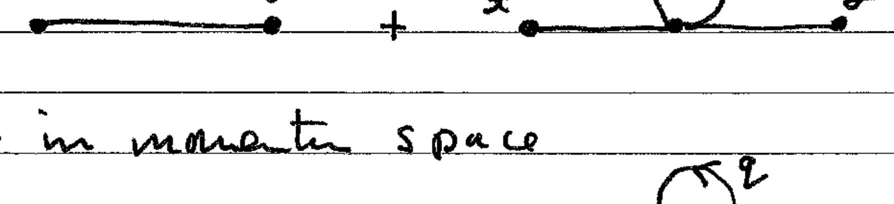
so $D_F(0) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon}$

This is divergent, we will come back to that later but it is our first example of a u.v. divergence in a ~~to~~ quantum loop; we can regularize it by introducing a high momentum cut-off $\Lambda$

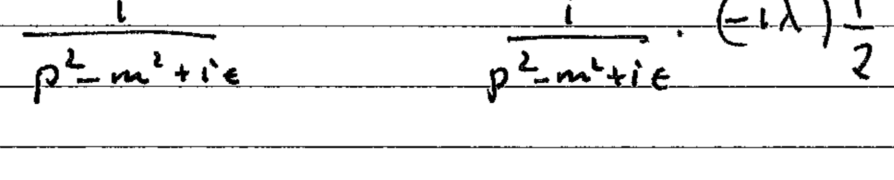
$$\int D_F(x-x')\, D_F(y-x')\, d^4x' = \int\int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} \frac{i}{q^2-m^2+i\epsilon} e^{-ip\cdot(x-x')} e^{-iq\cdot(y-x')} d^4x'$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \cdot \frac{i}{p^2-m^2+i\epsilon} \cdot \frac{i}{p^2-m^2+i\epsilon}$$

There is a diagramatic representation of this




or in momentum space




$$\frac{i}{p^2-m^2+i\epsilon} \qquad \frac{i}{p^2-m^2+i\epsilon} \cdot (-i\lambda) \frac{1}{2} \left[ \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2-m^2+i\epsilon} \right] \times \frac{i}{p^2-m^2+i\epsilon}$$

Note that the denominator has a graph representation which contains only closed loops and no external lines. At the order we have been working it is just  "bubblegraph"
Secondly, no matter what order we work to the effect of the denominator will be to cancel off graphs generated by the numerator which contain bubble graphs as sub-graphs. For example in

$$\langle \Omega | T\, \phi_{Full}(x)\, \phi_{Full}(y)\, | \Omega \rangle$$

$$= \frac{\langle 0 | T\, \phi(x)\, \phi(y) \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}\left( -i \frac{\lambda}{4!} \int d^4x'\, \phi(x')^4 \right)^k | 0 \rangle}{\langle 0 | T \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}\left( -i \frac{\lambda}{4!} \int d^4x'\, \phi(x')^4 \right)^k}$$

if we contract $\phi(x) + \phi(y)$ then they are removed from the ~~number~~ numerator and we get

$$\langle 0 | T\, \phi(x) \phi(y) | 0 \rangle \times \frac{\langle 0 | T \sum_{k=0}^{\infty} \ldots | 0 \rangle}{\langle 0 | T \sum_{k=0}^{\infty} \ldots | 0 \rangle}$$

It is easy to show that this always happens and we are left with graphs that are connected to external space-time points – in this case $x$ and $y$.

The next simplest case consists of four "external" fields so

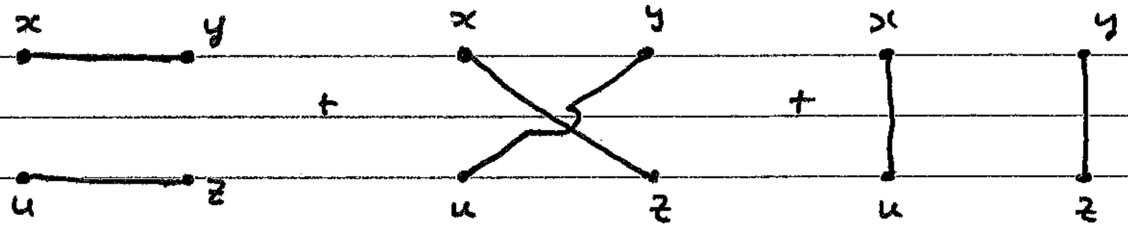
$$\langle \Omega | T \phi_{Full}(x)\, \phi_{Full}(y)\, \phi_{Full}(z)\, \phi_{Full}(u) | \Omega \rangle$$

$$= \frac{\langle 0 | T \phi(x)\phi(y)\phi(z)\phi(u) \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-i\lambda}{4!}\int d^4x'\, \phi(x')^4\right)^k | 0 \rangle}{\langle 0 | T e^{-\frac{i\lambda}{4!}\int d^4x'\, \phi(x')^4} | 0 \rangle}$$

The order $\lambda^0$ term is just

$$\langle 0 | T \phi(x)\, \phi(y)\, \phi(z)\, \phi(u) | 0 \rangle$$

$$= D_F(x-y)\, D_F(z-u) + D_F(x-z)\, D_F(y-u) + D_F(x-u)\, D_F(y-z).$$




and as previously discussed all the bubble graphs cancel out.

The order $\lambda$ term has two sorts of term in the denominator. ~~Thus~~

a) Two external $\phi$'s are contracted and the other two are contracted with an interaction term. This will generate

$$\langle 0|T\,\phi(x)\phi(y)|0\rangle\,\langle 0|T\,\phi(z)\phi(u)\left(-\frac{i\lambda}{4!}\int d^4x'\,\phi(x')^4\right)|0\rangle$$

$$\times\ \langle 0|T\sum_{k=1}^{\infty}\frac{1}{k!}\left(-\frac{i\lambda}{4!}\int d^4x'\,\phi(x')^4\right)^{k-1}\times k\,|0\rangle$$

$$\times\ \frac{1}{\langle 0|T\,e^{-i\lambda/4!\int d^4x'\phi(x')^4}|0\rangle}$$

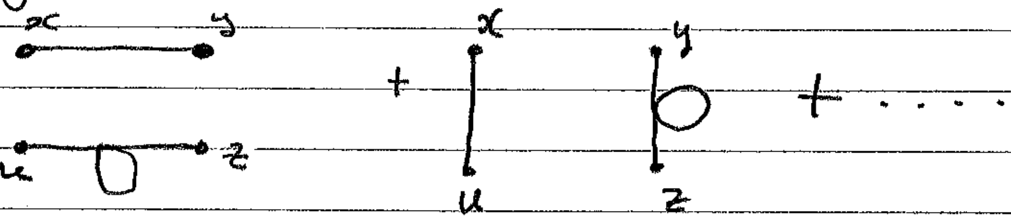
because there are k ways of picking the factor that goes in

Note the bubble terms cancel and we are left with

$$= \langle 0|T\,\phi(x)\phi(y)|0\rangle\,\langle 0|T\,\phi(z)\phi(u)\left(-\frac{i\lambda}{4!}\int\ldots\right)|0\rangle$$

~~Diag~~ + permutation of $x, y, z, u$

Diagrammatically we get




Altogether 6 terms, all disconnected

b) The really interesting term is when all four external fields contract with the interaction term. By the same argument as in a) the bubbles cancel and we are left with

$$-\frac{i\lambda}{4!}\int d^4x'\ \underset{\times 4}{D_F(x-x')}\ \times\ \underset{\times 3}{D_F(y-x')}\ \times\ \underset{\times 2}{D_F(z-x')}$$

$$\times\ \underset{\times 1}{D_F(u-x')}$$

$$= -i\lambda \int d^4x' \int \frac{d^4p_1}{(2\pi)^4} \frac{i}{p_1^2 - m^2 + i\epsilon} e^{-ip_1\cdot(x-x')}$$

$$\times \int \frac{d^4p_2}{(2\pi)^4} \frac{i}{p_2^2 - m^2 + i\epsilon} e^{-ip_2\cdot(y-x')}$$

$$\times \int \frac{d^4p_3}{(2\pi)^4} \frac{i}{p_3^2 - m^2 + i\epsilon} e^{-ip_3\cdot(z-x')}$$

$$\times \int \frac{d^4p_4}{(2\pi)^4} \frac{i}{p_4^2 - m^2 + i\epsilon} e^{-ip_4\cdot(u-x')}$$

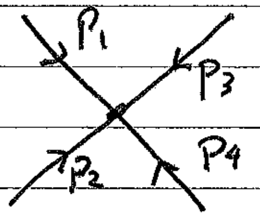
$$= -i\lambda \int \frac{d^4p_1}{(2\pi)^4} e^{-ip_1\cdot x} \int \frac{d^4p_2}{(2\pi)^3} e^{-ip_2\cdot y} \int \frac{d^4p_3}{(2\pi)^3} e^{-ip_3\cdot y}$$

$$\times \int \frac{d^4p_4}{(2\pi)^3} e^{-ip_4\cdot u} \, \delta^4(p_1+p_2+p_3+p_4)$$

4 momentum conserving
$\delta$ function

$$\times \prod_{k=1}^{4} \left( \frac{i}{p_k^2 - m^2 + i\epsilon} \right)$$

In momentum space




$$-i\lambda \times \prod_{k=1}^{4} \frac{i}{p_k^2 - m^2 + i\epsilon} \times \delta^4(p_1+p_2+p_3+p_4)$$

vertex

Feynman propagator for each leg

You can see that generates a real interaction between particles.

You can also see that there is a set of rules developing for writing down the answer without having to go through the Wick expansion every time.