Lecture 11 Feynman Rules and Scattering Matrix Elements

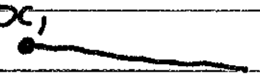
As we saw in the last lecture there is a graphical representation of our results for

$$G_n(x_1,\dots,x_n) = \langle \Omega | T \, \phi(x_1) \dots \phi(x_n) | \Omega \rangle_C$$ ← for connected

1 Draw all possible diagrams with n external lines and k vertice $\times$ for order λ^k contribution. Omit all diagrams that contain vacuum bubbles. It is sufficient to compute connected diagrams; so omit all diagrams that do not have a continuous path between all the external space time points. If we need them disconnected diagrams can be reconstructed from connected ones of lower n

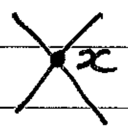
2 For diagrams in coordinate space

i) label external vertices



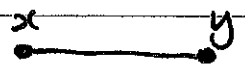
factor 1

ii Label vertices



factor $-i\lambda \int d^4x$

iii for each line

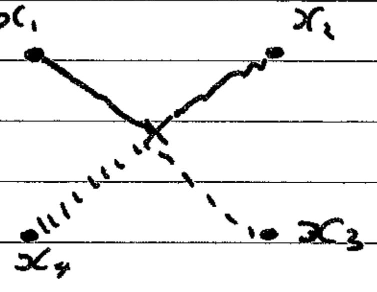


factor $D_F(x-y)$

iv multiply by the combinatorial factor

– or divide by the symmetry factor which is equivalent – arising from the number of ways the Wick contraction can be done to get the ~~graph~~ diagram.

For example



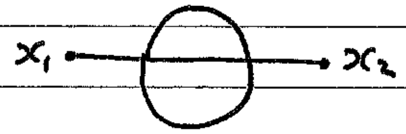
this comes from $H_{Int} = \frac{\lambda}{4!}\phi^4$

$$\frac{1}{4!} \times 4 \times 3 \times 2 \times 1$$

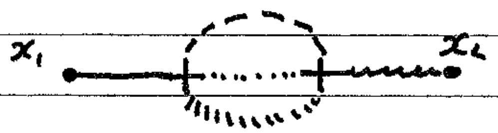
$$= 1$$

Combinatorial factor = 1 (Symmetry factor = 1)

Second example



from H_{int} from $e^{\ldots H_{int}} = 1 + \ldots H_{int} + \frac{1}{2!}(H_{int})^2$



$$\frac{1}{4!}\,\frac{1}{4!}\,\frac{1}{2!} \times 8 \times 4 \times 3 \times 2 \times 1$$

$$= \frac{1}{4!}\,\frac{1}{4!}\,\frac{1}{2!} \times 8 \times 4!$$

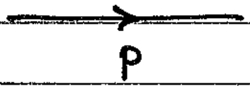
$$= \frac{1}{6}$$

3 Alternatively we can give rules in momentum space which is usually more convenient

$$G_n(x_1, x_2, \dots x_n) = \prod_{i=1}^{n} \int \frac{d^4 k_i}{(2\pi)^n} \, e^{-i \sum_{j=1}^{n} k_j \cdot x_j} \, \delta^4\left(\sum_{\ell=1}^{n} k_\ell\right)$$
$$\times \, \bar{G}_n(k_1, \dots k_n)$$

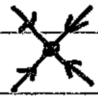
The rules for $\bar{G}_n$ are to label graphs with momenta such that momenta are conserved and then

i) for each line with momentum p



factor $\dfrac{i}{p^2 - m^2 + i\epsilon}$

ii) for each vertex

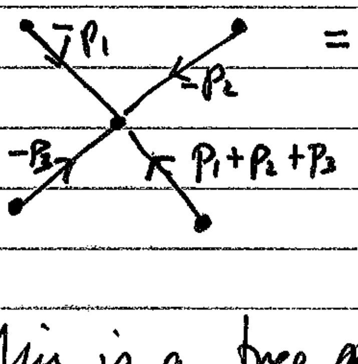


factor $-i\lambda$

iii) for each undetermined momentum $\displaystyle\int \frac{d^4 p}{(2\pi)^4}$

iv) multiply by the combinatorial factor

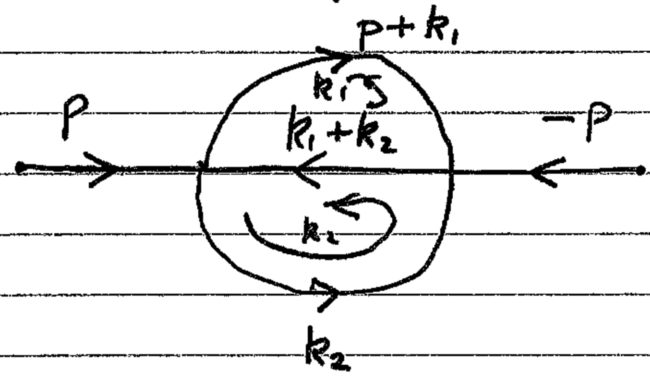
For example



$$= \frac{i}{p_1^2 - m^2 + i\epsilon} \, \frac{i}{p_2^2 - m^2 + i\epsilon} \, \frac{i}{p_3^2 - m^2 + i\epsilon}$$
$$\times \frac{i}{(p_1 + p_2 + p_3)^2 - m^2 + i\epsilon} \times -i\lambda$$

This is a tree diagram – it has no loops.

Second example



$$= \frac{1}{6} \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{p^2 - m^2 + i\epsilon} (-i\lambda)^2$$

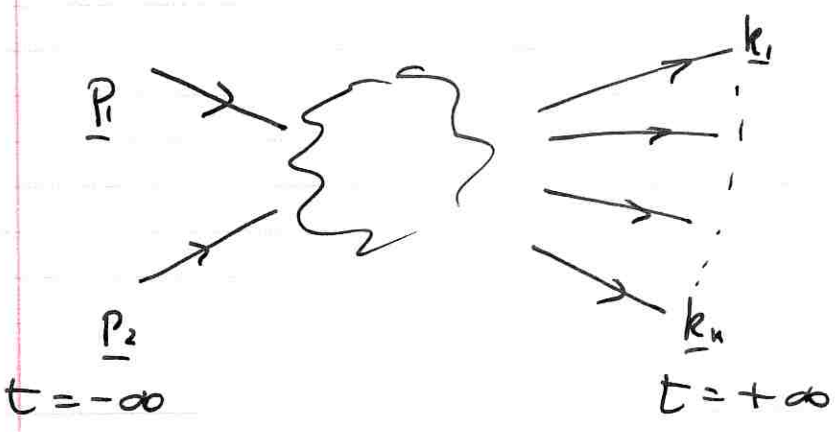
$$\int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \times \frac{i}{(p+k_1)^2 - m^2 + i\epsilon} \times \frac{i}{(k_1+k_2)^2 - m^2 + i\epsilon}$$

$$\times \frac{i}{k_2^2 - m^2 + i\epsilon}$$

Ultimately we will learn how to do momentum integrals but first we will work out how to relate these Green's functions to the matrix elements for scattering processes.

Scattering and Cross-sections

Ultimately we must relate the objects we can calculate in a QFT to things we can measure in experiments. This is not totally straightforward and remains a present-day research topic – for example in using QCD to predict the outcome of experiments at the LHC. The most ubiquitous process is the scattering of two particles to a multiparticle final state



and we will use this as an example. The basic experimental quantity is the cross-section. Working in a frame with a stationary target-particle and an incident beam

the basic experimental quantity is the cross-section

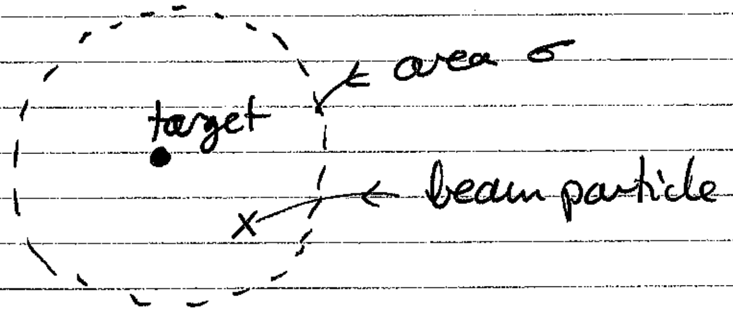
$$\sigma = \frac{\text{Rate of production of final state}}{\text{Beam flux} \times \text{Target Number}}$$

$$= \frac{\text{Number of events producing final state in time } T}{\text{Number of incident particles/unit area in time } T \times \text{Target Number}}$$

$$= \frac{\text{Probability of producing final state in time } 2\tau}{\text{Probability of 1 incident particle/unit area in time } 2\tau \times \text{Target Number}}$$

Obviously the first form is appropriate for analyzing experiment, and the second more useful for a theoretical calculation. Note that

- σ has dimensions of <u>area</u> – you can think of it as the effective area around the target within which the beam particle has to fall for an interaction to happen



- and that σ depends on intrinsic physics of the process, not on extrinsic characteristics of the experiment.

<u>Calculating σ</u>

We must be very careful about normalizations in this calculation so we will go back to the lattice picture in which we have taken the continuum limit $(a \to 0, Na = L)$ but not the infinite volume limit so L is finite, and we will work with the time interval $-\tau$ to $+\tau$. In this picture $a_{\underline{p}}^{+}|0\rangle$ is normalized to 1. (Recall that $[a_{\underline{p}}^{+}, a_{\underline{q}}] = -\delta_{\underline{p}+\underline{q},0}$.) So the correct matrix element for the process is,

$$\overbrace{\langle \underline{k}_1, \dots \underline{k}_n, t=+\tau}^{\text{"out-state"}} | \overbrace{\underline{p}_1, \underline{p}_2, t=-\tau \rangle_L}^{\text{"in-state"}}$$

$$= \langle \Omega_\tau | \left(\prod_{i=1}^{n} a_{\underline{k}_i} \right)_{+\tau} \left(a^+_{\underline{p}_1} a^+_{\underline{p}_2} \right)_{-\tau} |\Omega_\tau\rangle_{L(\text{attice})}$$

where we have assumed that at very large τ, ϕ behaves like a free field and the probability is

$$P = \prod_{i=1}^{n} \left(\sum_{\underline{k}_i \in B_3} \right) \left| \langle \Omega_\tau | \left(\prod_{i=1}^{n} a_{\underline{k}_i} \right)_{+\tau} \left(a^+_{\underline{p}_1} a^+_{\underline{p}_2} \right)_{-\tau} |\Omega_\tau\rangle_L \right|^2$$

$$= \prod_{i=1}^{n} \left(\frac{1}{L^3} \sum_{\underline{k}_i \in B_3} \frac{1}{2E_{\underline{k}_i}} \right) \frac{1}{L^6\, 2E_{\underline{p}_1} 2E_{\underline{p}_2}} \left| M_{fi}(\tau, L) \right|^2$$

where we define

$$M_{fi}(\tau, L) = \langle \Omega_\tau | \left(\prod_{i=1}^{n} a_{\underline{k}_i} \sqrt{2E_{\underline{k}_i}}\, L^{3/2} \right)_{\tau} \left(\sqrt{2E_{\underline{p}_1}}\, L^{3/2} a^+_{\underline{p}_1} \sqrt{2E_{\underline{p}_2}}\, L^{3/2} a^+_{\underline{p}_2} \right)_{-\tau}$$

$$\times |\Omega_\tau\rangle_L \quad (1)$$

When we calculate $M_{fi}(\tau, L)$ we find energy and momentum conserving Kronecker δ's so that

$$M_{fi}(\tau, L) = \underset{\text{convention}}{i}\; L^3 \overset{\text{momentum}}{\delta_{\sum_i \underline{k}_i - \underline{p}_1 - \underline{p}_2, 0}}\; 2\tau\, \overset{\text{energy}}{\delta_{\sum_i E_{\underline{k}_i} - E_{\underline{p}_1} - E_{\underline{p}_2}, 0}}$$

$$\times \hat{M}_{fi}(\tau, L) \quad (1)$$

which defines $\hat{M}_{fi}(\tau, L)$. Note that squaring a Kronecker δ gives the same δ back so

$$|M_{fi}(\tau, L)|^2 = L^6 \delta_{\sum_i \underline{k}_i - \underline{p}_1 - \underline{p}_2, 0} (2\tau)^2 \delta_{\sum_i E_{\underline{k}_i} - E_{\underline{p}_1} - E_{\underline{p}_2}, 0}$$

$$\times |\hat{M}_{fi}(\tau, L)|^2$$

and

$$P = \prod_{i=1}^{n} \left(\frac{1}{L^3} \sum_{\underline{k}_i \in B_3} \frac{1}{2E_{\underline{k}_i}} \right) \times \frac{2\tau}{L^3\, 2E_{\underline{p}_1} 2E_{\underline{p}_2}} \times L^3 \delta_{\sum_i \underline{k}_i - \underline{p}_1 - \underline{p}_2, 0}$$

$$\times 2\tau\, \delta_{\sum_i E_{\underline{k}_i} - E_{\underline{p}_1} - E_{\underline{p}_2}} \left| \hat{M}_{fi}(\tau, L) \right|^2$$

To get σ we need

1. Target number. This is 1 as we are dealing with states normalized to 1.

2. Probability of 1 incident particle per unit area per unit time. We are dealing with an isolated particle whose wave function is $\frac{e^{-ip\cdot x}}{L^{3/2}}$ so it is normalized to 1 in the volume L^3. The flux must come from the KG current normalised so that it gives the Schrodinger probability flux in the non-relativistic limit

$$\underline{j} = \frac{-i}{2m}\left(\psi^* \underline{\nabla}\psi - (\underline{\nabla}\psi^*)\psi\right) = \frac{1}{L^3}\frac{2\underline{p}}{2m} = \frac{\underline{v}}{L^3}$$

(note that this has the correct dimensions $L^{-2}T^{-1}$).

So

$$\sigma = \frac{P}{\frac{|\underline{v}|}{L^3} \times 2\tau \times 1}$$

$$= \prod_{i=1}^{n}\left(\frac{1}{L^3}\sum_{\underline{k}_i \in B_3}\frac{1}{2E_{\underline{k}_i}}\right)\frac{1}{2E_{\underline{p}_1}2E_{\underline{p}_2}|\underline{v}|} \times L^3\delta_{\sum_i \underline{k}_i - \underline{p}_1 - \underline{p}_2, 0}$$

$$\times\ 2\tau\,\delta_{\sum_i E_{\underline{k}_i} - E_{\underline{p}_1} - E_{\underline{p}_2}}\ \left|\hat{M}_{fi}(\tau, L)\right|^2$$

where $\underline{v}$ is the relative velocity of beam particle and target particle. Now we have σ in a form where we can take $L\to\infty$ and $\tau\to\infty$ giving

$$\sigma = \prod_{i=1}^{n}\left(\int\frac{d^3\underline{k}_i}{(2\pi)^3}\frac{1}{2E_{\underline{k}_i}}\right)\frac{1}{2E_{\underline{p}_1}2E_{\underline{p}_2}|\underline{v}|}(2\pi)^4\delta^4\left(\sum_i k_i - p_1 - p_2\right)$$

$$\times \lim_{\substack{\tau\to\infty\\ L\to\infty}} \left|\hat{M}_{fi}(\tau, L)\right|^2$$

LSZ Formalism

Now we have to deal with matrix element. Recall from (1) that

$$\langle \Omega_\tau | \left(\prod_{i=1}^{n} a_{\underline{k}_i} \sqrt{2E_{\underline{k}_i}}\, L^{3/2} \right)_\tau \left(\prod_{j=1}^{2} a^\dagger_{\underline{p}_j} \sqrt{2E_{\underline{p}_j}}\, L^{3/2} \right)_{-\tau} | \Omega_\tau \rangle_{\text{Lattice}}$$

$$= i\, L^3\, \delta_{\sum_i \underline{k}_i - \underline{p}_1 - \underline{p}_2, 0}\; 2\tau\, \delta_{\sum_i E_{k_i} - E_{p_1} - E_{p_2}} \times \hat{M}_{fi}(\tau, L).$$

The ~~available~~ annihilation and creation operators are still lattice operators and are related to the continuum ones by $\left(L^{3/2} a_{\underline{k}}\right)_{\text{lattice}} = L^{3/2} L^{-3/2} \left(a_{\underline{k}}\right)_{\text{continuum}}$ so we can take the $\tau, L \to \infty$ limit to get the continuum, ∞ volume result

$$\lim_{\tau \to \infty} \langle \Omega_\tau | \left(\prod_{i=1}^{n} a_{\underline{k}_i} \sqrt{2E_{\underline{k}_i}} \right)_\tau \left(\prod_{j=1}^{2} a^\dagger_{\underline{p}_j} \sqrt{2E_{\underline{p}_j}} \right)_{-\tau} | \Omega_\tau \rangle_{\text{continuum}}$$

$$= i(2\pi)^4 \delta^4\left(\sum_i k_i - p_1 - p_2 \right) \lim_{\substack{\tau \to \infty \\ L \to \infty}} \hat{M}_{fi}(\tau, L) \qquad (*)$$

so we need to compute the quantity on the l.h.s.

As noted earlier we assume that as $t \to \pm\infty$ the full field ϕ creates or destroys a particle which may interact with itself subsequently before interacting with other particles

However the full field is not in general a free field. As we showed a while back it is related to the interaction representation field ϕ_I by the U operator. So formally ϕ has a perturbation expansion in which at a given time

$$\phi = \underset{\substack{\uparrow \\ \text{creates single} \\ \text{particles}}}{\phi_{free}} Z(\lambda)^{1/2} + \underset{\substack{\uparrow \\ \text{creates eg multiple} \\ \text{particles and states} \\ \text{at } \mathcal{O}(\lambda)}}{\phi_{not\ free}}$$

$$Z(\lambda) = 1 + \mathcal{O}(\lambda)$$

is, as usual, a formal expansion in powers of λ

The normalization of fields is fixed by the canonical commutator at equal time which must ultimately control $Z(\lambda)$. For <u>one</u> particle states we have

$$\langle 1 \text{ particle state} | \phi | \Omega \rangle = Z(\lambda)^{1/2} \langle 1 \text{ particle state} | \phi_{free} | \Omega \rangle \quad (**)$$

In practice we compute $Z(\lambda)$ from the quantum corrections to the two point function in the Feynman graph expansion. We will come back to that.

It is easy to check that for a free field

$$\int d^3\underline{x}\, e^{-iq\cdot x} \overleftrightarrow{\partial_0} \phi_{free} = i\left(a^+_{\underline{q}} \sqrt{2E_{\underline{q}}}\right)_t$$

$$\int d^3\underline{x}\, e^{iq\cdot x} \overleftrightarrow{\partial_0} \phi_{free} = -i\left(a_{\underline{q}} \sqrt{2E_{\underline{q}}}\right)_t$$

Using these ~~into~~ results, + (**) on page 11.10, we find that the matrix element in (*) on page 11.9 is given by

$$\left(-iZ^{-1/2}\right)^{n+2} \lim_{\tau\to\infty} \langle\Omega_\tau| T \prod_{i=1}^{n}\left(\int d^3\underline{x}_i\, e^{ik_i\cdot x_i} \overleftrightarrow{\partial_{0i}} \phi\right)_\tau$$
$$\times \prod_{j=1}^{2}\left(\int d^3\underline{y}_j\, e^{-ip_j\cdot x_j} \overleftrightarrow{\partial_{0j}} \phi\right)_{-\tau} |\Omega_\tau\rangle$$

We have inserted the T product to facilitate the next step; at this point it has had no effect and is trivially correct. Now we note that

$$f(\tau) = f(-\tau) + \int_{-\tau}^{\tau} \frac{\partial f}{\partial t}\, dt$$

Applying this to $(a_{\underline{k}})_\tau$ we have

$$\sqrt{2E_{\underline{k}}}\,(a_{\underline{k}})_\tau = \sqrt{2E_{\underline{k}}}\,(a_{\underline{k}})_{-\tau} + +i\int_{-\tau}^{\tau} \frac{\partial}{\partial t}\left(\int d^3\underline{x}\, e^{ik\cdot x} \overleftrightarrow{\partial_0} \phi\right) dt$$

The $(a_{\underline{k}})_{-\tau}$ will annihilate $|\Omega_\tau\rangle$ unless $\underline{k} = \underline{p}_i$ in which case there is no scattering (a particle of momentum $\underline{p}_i$ just goes straight through) so we can drop the term.

~~The $a_{\underline{k}_i}$ will annihilate the r.h. $|0\rangle$ unless $\underline{k}_i = \underline{p}_r$ in which case there is no scattering and we can drop it.~~

We can repeat this exercise with all the other a's and $a^\dagger$'s and we are left with.

$$Z^{-\frac{n+2}{2}} (-i)^{n+2} \lim_{T\to\infty} \langle \Omega_T | T \prod_{i=1}^{n} \left(\int d^4x_i \; \partial_{0_i} \left(e^{ik_i\cdot x_i} \overleftrightarrow{\partial}_{0_i} \phi \right) \right)$$
$$\times \prod_{j=1}^{2} \left(\int d^4y_j \; \partial_{0_j} \left(e^{-ip_j\cdot x_j} \overleftrightarrow{\partial}_{0_j} \phi \right) \right) |\Omega_T\rangle_C$$

where C is to remind us that we want only connected diagrams.
Now

$$\int d^4x \; \partial_0 \left(e^{ik\cdot x} \partial_0 \phi - (\partial_0) e^{ik\cdot x} \phi \right)$$
$$= \int d^4x \left(e^{ik\cdot x} \partial_0^2 \phi - (\partial_0^2 e^{ik\cdot x}) \phi \right)$$
$$= \int d^4x \; e^{ik\cdot x} \partial_0^2 \phi + E_{\underline{k}}^2 e^{ik\cdot x} \phi$$
$$= \int d^4x \; e^{ik\cdot x} \partial_0^2 \phi + \left((-\underline{\nabla}^2 + m^2) e^{ik\cdot x} \right) \phi$$

integrating by parts $d^3\underline{x}$ and discarding terms at spatial ∞

$$= \int d^4x \; e^{ik\cdot x} (\partial^\mu \partial_\mu + m^2) \phi$$

so finally we get in the interaction representation

$$Z^{-\frac{n+2}{2}} (-i)^{n+2} \prod_{i=1}^{n} \left(\int d^4x_i \, e^{ik_i\cdot x_i} (\Box_i + m^2) \right) \prod_{j=1}^{2} \left(\int d^4y_j \, e^{-ip_j\cdot x_j} (\Box_j + m^2) \right).$$

$$\langle 0 | T \begin{pmatrix} \phi_I(x_1) \ldots \phi_I(x_n) \;\; \phi_I(y_1)\, \phi_I(y_2) \\ \times \exp\left(-i \int_{-\infty}^{\infty} H_{int}(\phi_I(t))\, dt \right) \end{pmatrix} | 0 \rangle_{NV,C}$$

where NV, C is to remind us that we exclude vacuum bubbles, and consider only connected diagrams.

We can see immediately by inserting the form $\ddagger$ on page 11.3 that the effect of the $\Box + m^2$ operators is to introduce a factor $(-p^2+m^2)(-iZ$ on each external line of a diagram. This cancels off the $\frac{i}{p^2-m^2+i\epsilon}$ factors on the external legs.

The ~~Feyn~~ quantity $\mathcal{M}$ that we need to calculate the cross-section is therefore computed from amputated graphs. The momentum space rules are exactly the same as before except that there is no $\frac{i}{p^2-m^2+i\epsilon}$ factor for external lines.