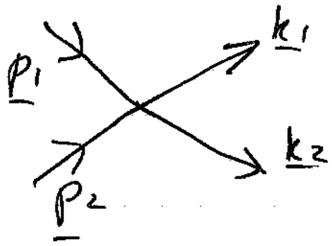
Lecture 12 The Scalar QFT, Tree graphs and One Loop

Scalar Scattering Processes at Tree graph level

The simplest scattering process in our scalar field theory is $2 \to 2$. At tree ~~graph~~ level there is only one diagram



and the amputated Green's function is simply $(-i\lambda)$

The cross section is given by

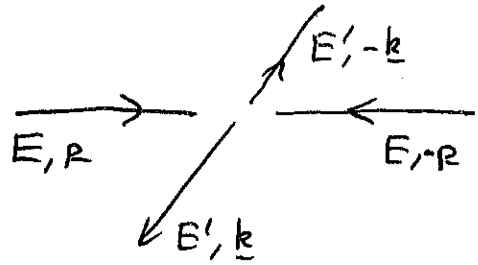
$$\sigma = \frac{1}{2!}\int \frac{d^3\underline{k}_1}{(2\pi)^3}\frac{1}{2E_{\underline{k}_1}}\int\frac{d^3\underline{k}_2}{(2\pi)^3}\frac{1}{2E_{\underline{k}_2}}\frac{1}{4E_{\underline{p}_1}E_{\underline{p}_2}|\underline{v}|}$$
$$\times\ (2\pi)^4\ \delta^4(k_1+k_2-P)\ \times\ \lambda^2$$

where we have defined $P = p_1+p_2$ and it is usual to define a quantity $s = P^\mu P_\mu$

We can integrate out $d^3\underline{k}_2$ which gives

$$\sigma = \frac{\lambda^2}{2!}\frac{1}{4E_{\underline{p}_1}E_{\underline{p}_2}|\underline{v}|}\int\frac{d^3\underline{k}_1}{(2\pi)^3}\frac{1}{2E_{\underline{k}_1}\,2E_{\underline{P}-\underline{k}_1}}2\pi\,\delta(E_{\underline{k}_1}+E_{\underline{P}-\underline{k}_1}-\underbrace{E}_{\text{total incident energy}})$$

It is extremely tiresome to do the rest of the integrals in the general case but it is easy to do the calculation in the centre of mass frame in the relativistic limit when $E_{\underline{k}} = |\underline{k}|$ etc. The kinematics looks like



$, \quad s = 4E^2,$

and $|\underline{v}| = 1$ (ie speed of light)

Then we get

$$\sigma = \frac{\lambda^2}{2s} \int \frac{k_1^2\, dk_1\, d\Omega_{\underline{k}_1}}{(2\pi)^3} \quad \frac{1}{4k_1^2} \cdot 2\pi\, \delta(2k_1 - 2E)$$

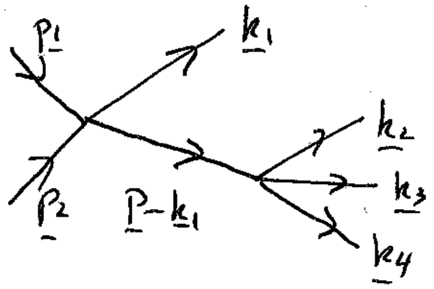
solid angle

$$= \frac{\lambda^2}{2s} \; \frac{1}{(2\pi)^3} \; \frac{4\pi}{4} \quad 2\pi \cdot \frac{1}{2}$$

$$= \frac{\lambda^2}{16\pi s}$$

Note that the cross-section falls off as $\frac{1}{s}$. This is generic behaviour at very high energies when particle masses are insignificant.

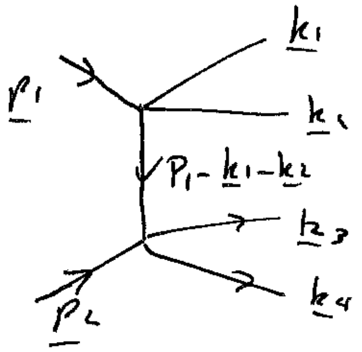
The independence of the matrix element from the particle momenta is peculiar to lowest order $2 \to 2$ scattering. We can see this immediately by examining $2 \to 4$ scattering. The graphs are



Note there are 4 different diagrams depending on whether $\underline{k}_1, \underline{k}_2, \underline{k}_3$ or $\underline{k}_4$ is the momentum attached to the first vertex and

Combinatorial factor

$\frac{1}{4!}\,\frac{1}{4!}\,\frac{1}{2!} \cdot 8.3.2\;\, 4\, 3 \times 2 = 1$



6 different diagrams $\binom{4}{2}$

and a combinatorial factor

$\frac{1}{4!}\,\frac{1}{4!}\,\frac{1}{2!} \;\; 8.4.3.2.3.2 = 1$

so altogether

$$M = \lambda^2 \sum_i \frac{i}{(P-k_i)^2 - m^2 + i\epsilon}$$
$$+ \lambda^2 \sum_{\langle ij \rangle} \frac{i}{(P_1 - k_i - k_j)^2 - m^2 + i\epsilon}$$

The integral over the final state is of course more complicated now. However it is straightforward to see that in the relativistic limit we will get

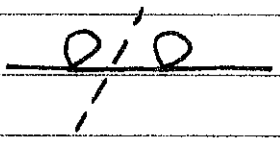
$$\sigma = \frac{\lambda^4}{s} \times \text{const}$$

- essentially the powers of s introduced by $\int \frac{d^3k_3}{2E_{k_3}} \int \frac{d^3k_4}{2E_{k_4}}$ are cancelled by the powers introduced by the propagators in ~~the~~ M. Also note that in the total cross-section the $\frac{1}{4!}$ works to decrease the value against the increasing number of graphs; in the end a full calculation ~~will~~ is necessary to get the constant factor.

Of course we can extend this exercise to 6, ... particle production. We won't do that here but in applications of eg QCD to collider physics such calculations are necessary, and obviously ~~[illegible]~~ complicated.

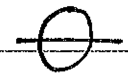
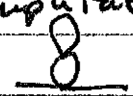
So

 is 1PI

but  is 1 Particle Reducible

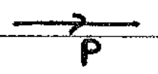
Now we define

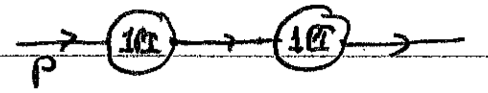
$$i\Sigma(p) = \text{Sum of all 1 PI diagrams with external lines amputated}$$

$=$  $+$  $+$ 

$=$ 

and then

$D(p) =$  $+$ 

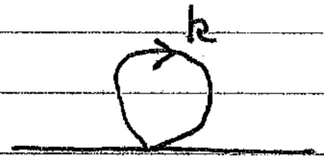
$+$  $+ \dots$

$$= \frac{i}{p^2 - m^2 + i\epsilon} + \frac{i}{p^2 - m^2 + i\epsilon}\, i\Sigma(p)\, D(p)$$

whence

$$D(p) = \frac{i}{p^2 - m^2 + \Sigma(p) + i\epsilon}$$

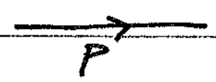
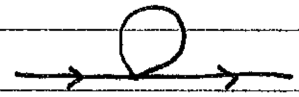
Now we will calculate $\Sigma^{(1)}(p)$, the $O(\lambda)$ contribution.



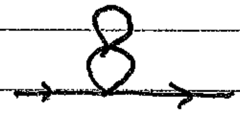
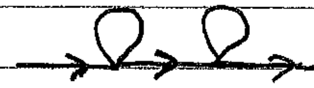
$$\Sigma^{(1)}(p) = -i\,(-i\lambda)\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2 - m^2 + i\epsilon}$$

One Loop Graphs

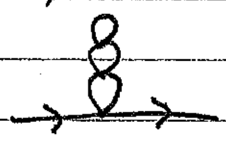
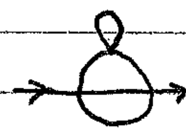
We have already met the 1 loop correction to the 2point function. To $O(\lambda)$ we have

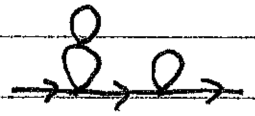
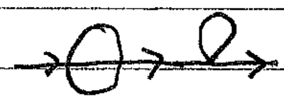
$D(p) =$  $+$ 

If we go to $O(\lambda^2)$ we get

$+$  $+$  $+$ 

to $O(\lambda^3)$

$+$  $+$  $+$  $+$ 2 others

$+$  $+$  $+$ 2 others.

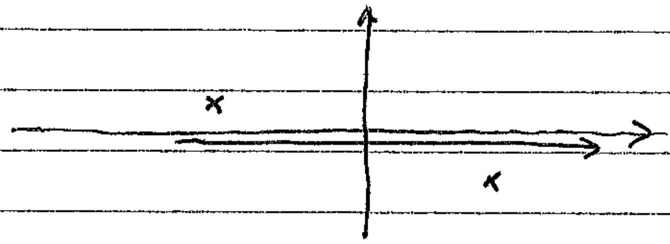
You can see that we generate two sorts of graphs

1. Graphs that really are new at the order we are working

2. Graphs that are combinations of lower order graphs strung together.

So we define One Particle Irreducible (1PI) Graphs to be those that cannot be separated into disjoint components by cutting a single <u>internal</u> line.

If the integration range is $-\infty$ to ∞ on all four components then the integral diverges. For example looking at the k_0 integral we have the usual situation



poles at $\pm\left(\left(\underline{k}^2+m^2\right)^{1/2}-i\epsilon\right)$. We can choose to close in the upper half plane to be left with

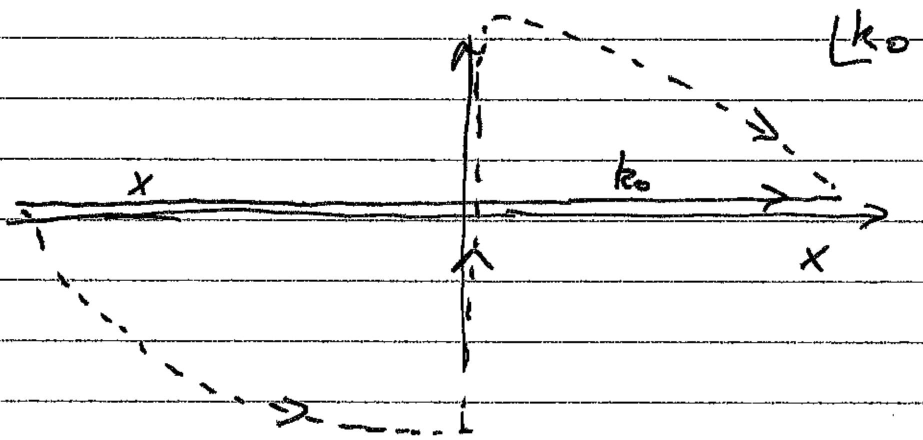
$$-i\lambda \int \frac{d^3\underline{k}}{(2\pi)^3}\,\frac{1}{2\pi}\,.\,2\pi i \times \frac{1}{-2\left(\underline{k}^2+m^2\right)^{1/2}}$$

$$= -\frac{\lambda}{2}\,\frac{1}{(2\pi)^3}\int_0^\infty \frac{k^2\,dk}{\sqrt{k^2+m^2}} \times 4\pi \text{ solid angle}$$

which is obviously infinite. This phenomenon occurs throughout quantum field theories; it is called an "ultra-violet divergence". We deal with it in a number of steps.

If you think back to our lattice formulation of QFT you will recall that the highest value of a momentum component was $\frac{2\pi}{a}$, not ∞, so that the theory in fact did have a momentum cut-off. We could reinstate that but it is convenient to proceed slightly differently. We have

$$I_0 = \int \frac{dk_0\, d^3\underline{k}}{(2\pi)^4} \frac{1}{k_0^2 - \underline{k}^2 - m^2 + i\epsilon}$$



Because no poles are enclosed the k_0 integral along the dashed path must give the same ~~integral~~ result as along the original path. The circular portions are at ∞ and don't contribute so writing $k_0 = iK$.

$$I_0 = i \int_{-\infty}^{\infty} \frac{dK}{2\pi} \int \frac{d^3\underline{k}}{(2\pi)^3} \frac{1}{-K^2 - \underline{k}^2 - m^2 + i\epsilon}$$

now the $i\epsilon$ is irrelevant and we have

$$I_0 = -i \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2}$$

where k_E is just a four component euclidean metric vector $k_E^2 = k_{E0}^2 + k_{E1}^2 + k_{E2}^2 + k_{E3}^2$

now we regularize the integral by imposing a cutoff on $|k_E|$ so that $|k_E| < \Lambda$ and we have

$$I_0(\Lambda) = -i \int_0^{\Lambda} \frac{k^3\, dk\, d\Omega_4}{(2\pi)^4 (k^2+m^2)} \qquad k = |k_E|$$

$$= -i \frac{2\pi^2}{(2\pi)^4} \cdot \frac{1}{2} \int_0^{\Lambda^2} \frac{x\, dx}{x+m^2} \qquad x = k^2$$

$$= -\frac{i}{16\pi^2} \int_0^{\Lambda^2} 1 - \frac{m^2}{x+m^2}\, dx.$$

$$= -\frac{i}{16\pi^2} \left(\Lambda^2 - m^2 \log \frac{\Lambda^2+m^2}{m^2} \right)$$

Note that

$$\cancel{I_n(\Lambda) = -i \int_0^{\Lambda} k^3\, dk\, d\Omega_4}$$

$$\cancel{I_n(\Lambda) =}$$

$$I_n \equiv \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k_0^2 - \underline{k}^2 - m^2 + i\epsilon)^{n+1}}$$

$$= \frac{+1}{n!} \left(\frac{\partial}{\partial m^2} \right)^n I_0 \qquad \text{eg } I_1 = \frac{i}{16\pi^2} \left(\log \frac{\Lambda^2+m^2}{m^2} - 1 + \frac{m^2}{\Lambda^2+m^2} \right)$$

which will be useful.

Now using our result for $I_0(\Lambda)$ we find that

$$\Sigma^{(1)}(p) = -\frac{\lambda}{16\pi^2} \left(\Lambda^2 - m^2 \log \frac{\Lambda^2+m^2}{m^2} \right)$$

and, dropping terms $O(1/\Lambda^2)$,

$$D(p) = \frac{i}{p^2 - m^2 - \frac{\lambda}{16\pi^2}\left(\Lambda^2 - m^2 \log \frac{\Lambda^2}{m^2}\right)}$$

so the one loop correction has induced a mass shift which is proportional to the cut-off. We will discuss the implications of this shortly.

Actually this result is slightly simpler than it would be at higher orders in the coupling. The diagram  will give a contribution to $\Sigma^{(2)}$ that is proportional to p^2 so in general

$$\Sigma(p) = (p^2 - m^2)(Z_{\Lambda p}^{-1} - 1) - Z^{-1}\delta m_\Lambda^2$$

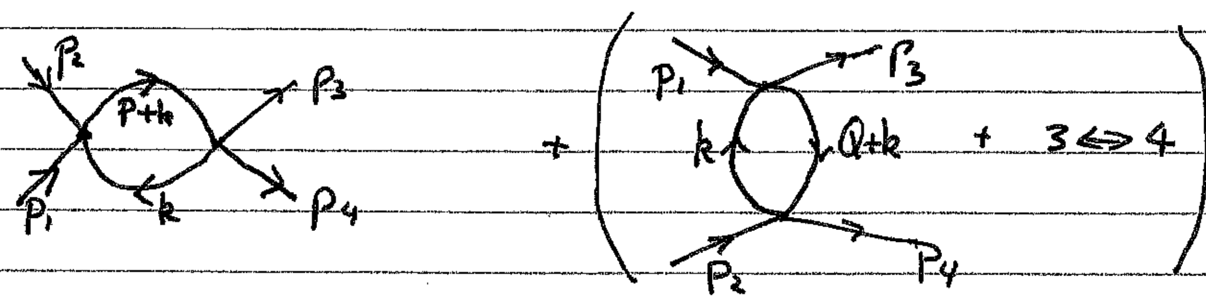
where $Z_{\Lambda p} = 1 + O(\lambda^2)$

and $\delta m^2_{\Lambda p} = \frac{\lambda}{16\pi^2}\left(\Lambda^2 - m^2 \log \frac{\Lambda^2}{m^2}\right) + O(\lambda^2)$

so that

$$D(p) = \frac{Z_{\Lambda p}}{p^2 - m^2 - \delta m_\Lambda^2}$$

The 1PI one loop correction to the 4-point function is given by the amputated diagrams



$P = p_1 + p_2$, $Q = p_1 - p_3 = p_2 - p_4$, which give

$R = p_1 - p_4 = p_2 - p_3$

$$\Gamma_4^{(1)} = \frac{(-i\lambda)^2}{2} \int \frac{d^4k\,(i)^2}{(2\pi)^4} \left\{ \frac{1}{(P+k)^2 - m^2 + i\epsilon} \, \frac{1}{k^2 - m^2 + i\epsilon} + \frac{1}{(Q+k)^2 - m^2 + i\epsilon} \, \frac{1}{k^2 - m^2 + i\epsilon} + \frac{1}{(R+k)^2 - m^2 + i\epsilon} \, \frac{1}{k^2 - m^2 + i\epsilon} \right\}$$

This can be simplified using the Feynman parameter trick

$$\frac{1}{AB} = \int_0^1 \frac{dx}{\left(Ax + B(1-x)\right)^2}$$

so for example

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(P-k)^2-m^2+i\epsilon} \frac{1}{k^2-m^2+i\epsilon}$$

$$= \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left(k^2 - 2k\cdot Px + P^2x - m^2 + i\epsilon\right)^2}$$

$$= \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left((k-Px)^2 - (m^2 - P^2x(1-x)) + i\epsilon\right)^2}$$

$$= \int_0^1 dx \frac{i}{16\pi^2} \left(\log \frac{\Lambda^2}{m^2 - P^2x(1-x)} - 1 + O\left(\frac{1}{\Lambda^2}\right)\right)$$

$$= \frac{i}{16\pi^2} \left(\log \Lambda^2/m^2 - 1 - \int_0^1 dx \log \frac{m^2 - P^2x(1-x)}{m^2} \right)$$

dropping $O\left(\frac{1}{\Lambda^2}\right)$ terms. Altogether

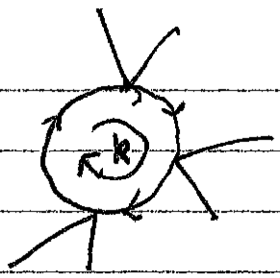
$$\Gamma_4^{(1)} = \frac{i\lambda^2}{32\pi^2} \left(3 \log \frac{\Lambda^2}{m^2} - 3 - \sum_{K=P,Q,R} \int_0^1 dx \log \frac{m^2 - K^2x(1-x)}{m^2} \right)$$

So altogether up to 1 loop order we have a 1PI vertex

$$\Gamma_4 = -i\lambda \left(1 - \frac{3\lambda}{32\pi^2} \log \frac{\Lambda^2}{m^2} + \frac{\lambda}{32\pi^2} \sum_{K=P,Q,R} F(K) \right)$$

where $F(K) = 1 + \int_0^1 dx \log \frac{m^2 - K^2x(1-x)}{m^2}$

The 1PI one loop functions for higher ~~orders are~~ numbers of external lines are u.v. finite. For example $n = 6$



There are three propagators so at high k the integral will look like

$$\lambda^3 \int d^4k \left(\frac{1}{k^2}\right)^3$$

which is a convergent integral. We will come back to the systematics of this in a while.

Effective and Renormalizable QFTs

There are (at least) two ways of thinking about the cut-off dependence that we have encountered. Both are valid, but historically the notion of renormalizable QFTs came long before the modern view of Effective QFTs. So in old books you will find much discussion of renormalizable and non-renormalizable QFTs with the implication that non-renormalizable QFTs are useless.