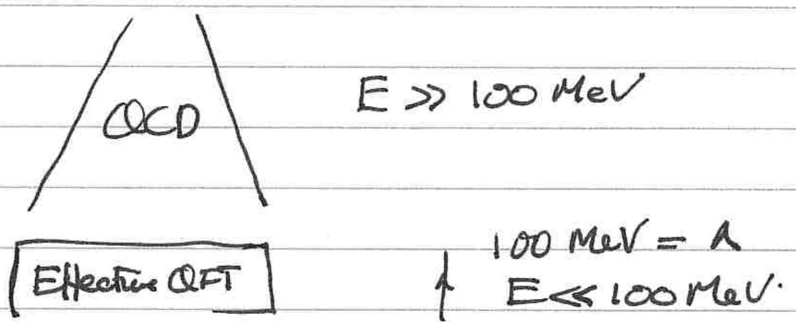
# Lecture 13 Effective + Renormalizable QFTs

## 1. Effective QFT

In this approach we accept the existence of the cut-off as a parameter in the theory. A good example is the theory of pions, $\pi^{\pm}\ \pi^{0}$ and nucleons p, n. At energies of $\ll 100$ MeV or so they behave as elementary scalars and fermions whose dynamics is well described by an effective QFT of scalars and fermion fields. At higher energies (shorter wavelengths) they are revealed to contain quarks and gluons and physics is described by QCD.




## 2. Renormalizable QFT

In this case we can consistently take the cut-off to infinity $\Lambda \to \infty$ without destroying the predictive capacity of the theory.

It's easiest to discuss this if we allow a slightly generalized Lagrangian density

$$\mathcal{L} = Z_\phi \left( \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 \right) + Z_\phi^2 \frac{\lambda}{4!} \phi^4$$

and then explicitly set $Z_\phi = 1$ in the calculations we have already made. 1) Now suppose we have a set of measurables, $Q_n$ for which we can calculate

$$Q_n = q_n \left( Z_\phi (=1), m, \lambda, \Lambda \right) \qquad (1)$$

measured ↑ a calculated function

2) Next choose three measurements, say $n = 1, 2, 3$ and solve the equations (1) for

$$\begin{aligned} m &= \bar{m} \left( Q_1, Q_2, Q_3, \Lambda \right) \\ \lambda &= \bar{\lambda} \left( Q_1, Q_2, Q_3, \Lambda \right) \\ Z_\phi &= \bar{Z}_\phi \left( Q_1, Q_2, Q_3, \Lambda \right) \end{aligned} \qquad (2)$$

$\bar{m}, \bar{\lambda}, \bar{Z}_\phi$ being some functions.

3) Then taking these solutions (2) and substituting back in (1) we find

$$Q_n = q_n \left( \bar{Z}_\phi (Q_1, Q_2, Q_3, \Lambda), \bar{m} (Q_1, Q_2, Q_3, \Lambda), \bar{\lambda} (Q_1, Q_2, Q_3, \Lambda), \Lambda \right) \qquad (3)$$

4) If the theory is renormalizable then the $\Lambda$ dependence drops out of the right hand side of (3) which reduces to

$$Q_n = \bar{q}_n(Q_1, Q_2, Q_3) \tag{4}$$

ie the theory relates measurable quantities through calculable relations (4). If the theory is <u>not</u> renormalizable this magic does not happen and we have to treat it as an effective field theory.

To implement this program it is convenient to note that because $Q_{1,2,3}$ are functions of $Z_\phi, m, \lambda, \Lambda$ we can write (3) as

$$Q_n = q_n\left(\hat{Z}_\phi(Z_\phi, m, \lambda, \Lambda), \hat{m}(Z_\phi, m, \lambda, \Lambda), \hat{\lambda}(Z_\phi, m, \lambda, \Lambda), \Lambda\right)$$

Then all $Q_n$ will come out independent of $\Lambda$ as before and we have a parametrization that is in terms of a Lagrangian and easy to use.

## Implementation

We start by calculating with

$$\hat{\mathcal{L}} = \hat{Z}_\phi \left( \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} \hat{m}^2 \phi^2 \right) - \hat{Z}_\phi^2 \frac{\hat{\lambda}}{4!} \phi^4$$

Then for the 2 point function we will get

$$D(p) = \frac{Z_{\Lambda p} \hat{Z}_\phi^{-1}}{p^2 - \hat{m}^2 - \delta m^2_\Lambda}$$

where

$$Z_{\Lambda p} = 1 + O(\lambda^2).$$

and $$\delta m^2_\Lambda = \frac{\hat{\lambda}}{16\pi^2} \left( \Lambda^2 - \hat{m}^2 \log \frac{\Lambda^2}{\hat{m}^2} \right) + O(\hat{\lambda}^2)$$

Now we need to choose associated observables. This choice is not unique but a simple method is to decide that at $p = p_0$ where $p_0^2 = m_{phys}^2$ $D(p)$ must take the form of a pole with residue 1. Then

$$D(p) = \frac{1}{p^2 - m_{phys}^2} + \text{regular}$$

This implies that

$$\hat{m}^2 + \delta m^2_\Lambda = m_{phys}^2 = p_0^2$$

and $$\hat{Z}_\phi = Z_{\Lambda p_0} \quad \left( = 1 + O(\hat{\lambda}^2) \right)$$

So

$$\hat{m}^2 = m_{phys}^2 - \frac{\hat{\lambda}}{16\pi^2} \left( \Lambda^2 - \hat{m}^2 \log \frac{\Lambda^2}{\hat{m}^2} \right) + O(\hat{\lambda}^2).$$

We can replace $\hat{m}$ and $\hat{\lambda}$ on the r.h.s. by $m_{phys}$ and $\lambda_{phys}$ (to be defined in a minute) making an error at $O(\lambda^2)$ so

$$\hat{m}^2 = m_{phys}^2 - \frac{\lambda_{phys}}{16\pi^2} \left( \Lambda^2 - m_{phys}^2 \log \frac{\Lambda^2}{m_{phys}^2} \right) + O(\lambda_{phys}^2).$$

Calculating with $\hat{\lambda}$ for the 4 point function we will get for the amputated matrix element

$$iM = \hat{\lambda}\,\hat{Z}_\phi^{\,2}\left(1 - \frac{\hat{\lambda}\,\hat{Z}_\phi^{\,2}}{32\pi^2}\left(3\log\frac{\Lambda^2}{\hat{m}^2} + \sum_{K=P,Q,R}\hat{F}(K)\right)\right) + O(\hat{\lambda}^3)$$

where $\hat{F}(K) = 1 + \int_0^1 dx\,\log\frac{\hat{m}^2 - K^2x(1-x)}{\hat{m}^2}$

The factor $\hat{Z}_\phi^{\,2}$ overall is cancelled by the $(Z^{-\frac{1}{2}})^{n+2}$ factor in the LSZ formalism. For the observable we choose the differential cross-section at a give set of momenta so that $iM \equiv \lambda_{phys}$ for those momenta

$$\lambda_{phys} = \hat{\lambda}\left(1 - \frac{\hat{\lambda}}{32\pi^2}\left(3\log\frac{\Lambda^2}{\hat{m}^2} + \sum_{K=P_0,Q_0,R_0}\hat{F}(K)\right)\right) + O(\hat{\lambda}^3)$$

(we have dropped $\hat{Z}_\phi$ altogether now as it doesn't contribute to $O\hat{\lambda}^2$). From this we get that

$$\hat{\lambda} = \lambda_{phys}\left(1 + \frac{\lambda_{phys}}{32\pi^2}\left(3\log\frac{\Lambda^2}{m_{phys}^2} + \sum_{K=P_0,Q_0,R_0}F_{phys}(K)\right)\right) + O(\lambda_{phys}^3)$$

$$F_{phys}(K) = 1 + \int_0^1 dx\,\log\frac{m_{phys}^2 - K^2x(1-x)}{m_{phys}^2}$$

To continue the process the next step would be to calculate to two loops and iterate the argument. But we would find, in this case, that carefully setting $\hat{Z}_\phi$, $\hat{m}^2$, $\hat{\lambda}$ at each order is enough to render all observables independent of $\Lambda$. So, for example, if we now compute the value of $iM$ at different momenta we find

$$iM = \lambda_{phys} + \frac{\lambda_{phys}^2}{32\pi^2} \sum_K \int_0^1 dx \, \log \frac{m_{phys}^2 - K^2 x(1-x)}{m_{phys} - K_{(0)}^2 x(1-x)}$$

where $K = P, Q, R$ and $K_{(0)} = P_0, Q_0, R_0$. It's interesting to note that as the energy goes up, ie $P, Q, R$ increase in magnitude, the magnitude of $M$, and hence the observed strength of the interaction also goes up. This leads us into the topic of the renormalization group which you'll meet next term.