Lecture 14

Dimension, Power Counting + Renormalizability

The renormalizability, or otherwise, of a QFT is a crucial characteristic. Here we will develop a simple set of criteria which, strictly speaking, show when a QFT cannot be renormalizable. Proving that the others are renormalizable to all orders in perturbation theory is a formidable task that we will not pursue.

To tell the order of divergence of diagrams we need some systematic way of counting up the powers of momentum which will be given by the Feynman rules. We can do this by dimensional analysis. Firstly let us go back to the Lagrangian and compute the dimensions of all the quantities appearing in a fairly general QFT:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - g\varphi\bar{\psi}\psi$$

$$- \frac{\eta}{3!}\varphi^3 - \frac{\lambda}{4!}\varphi^4 - \frac{\zeta}{6!}\varphi^6 - \epsilon(\bar{\psi}\psi)^2 \ldots$$

$i \int d^d x\, \mathcal{L}$ must be dimensionless (we have set $\hbar = 1$)

Look at the φ kinetic term $\frac{1}{2}(\partial_\mu \varphi)^2$

we must have $[\varphi] = 1$ where $[\ ]$ means in momentum (or mass) units

the mass term then gives $[m^2] = 2$ as we should expect

The ψ kinetic term gives

$$[\psi] = [\bar{\psi}] = \frac{3}{2}$$

$$[M] = 1.$$

The interaction terms $-g\varphi\bar{\psi}\psi \Rightarrow [g] = 0$

$$\eta\varphi^3 \Rightarrow [\eta] = 1$$

$$\lambda\varphi^4 \Rightarrow [\lambda] = 0$$

$$\zeta\varphi^6 \Rightarrow [\zeta] = -2$$

$$\epsilon(\bar{\psi}\psi)^2 \Rightarrow [\epsilon] = -2$$

The dimension of these terms is what determines whether or not a renormalization procedure will work. Note that the dimension of the coupling constant is given

by $d_v = 4 - \frac{3}{2} n_F - n_B$ | $n_{F,B}$ = number of fermion, boson fields in interaction term

Now define for a given graph the quantities

E_F external fermion lines
E_B external boson lines
I_F internal fermion lines
I_B " boson "
L loops
V vertices
$I = I_F + I_B$ total number of internal lines.

Remember that Feynman rules give an arbitrary momentum k to each internal line and a δ fn to each vertex. The number of independent loop integrals

$$L = I - (V-1)$$

← must be left over with one δ function conserving 4-momentum for the whole diagram.

To determine whether a diagram is likely to diverge let us consider rescaling the momenta+mass $k \to \lambda k$, $m \to \lambda m$ and count the power of λ generated by the Feynman rules

$$d^4k \to \lambda^4 d^4k$$
fermion prop $S_F \to \lambda^{-1} S_F$
boson " $G_F \to \lambda^{-2} G_F$

For a diagram we find λ^ω where

$$\lambda^\omega = (\lambda^4)^L (\lambda^{-1})^{I_F} (\lambda^{-2})^{I_B}$$

(This is only true for theories without derivative couplings to keep matters simple).

$$\therefore \quad \omega = 4L - I_F - 2I_B$$
$$= 3I_F + 2I_B - 4(V-1) \quad \ddagger$$

Further progress depends upon the nature of the vertices; then we can rewrite everything in terms of the numbers of external lines. Suppose there is only one type of vertex which has n_F fermion legs
n_B boson legs

and observe that each external line is attached to one vertex
" internal " " " " two vertices

$$\left.\begin{aligned} E_F + 2I_F &= Vn_F \\ E_B + 2I_B &= Vn_B \end{aligned}\right\}$$ solve for I_F and I_B and put back into expression ‡ for ω

$$\omega = \frac{3}{2}\left(Vn_F - E_F\right) + \left(Vn_B - E_B\right) - 4(V-1)$$

$$= -\frac{3}{2}E_F - E_B + 4 + V\left(\frac{3}{2}n_F + n_B - 4\right)$$

$$= -\frac{3}{2}E_F - E_B + 4 - Vd_V$$

where d_V is just the dimension of the coupling constant associated with the vertex. Now we can see how various Green's functions behave at high loop momenta.
ω is called the superficial degree of divergence

1) Start with φ^3 theory; $d_V = 1$. so

$$\omega = 4 - E_B - V$$

The one-point function $E_B = 1$ superficially diverges if $V = 1, 2, 3$. Therefore, looking at 1PI diagrams

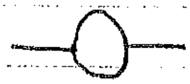
$V=1$  is divergent

$V=2$ there is no diagram

$V=3$  is divergent but has an embedded 2pt function 

 is not 1PI, + has an embedded one loop 1pt function

The two-pt function $E_B = 2$ diverges if $V = 1, 2$ but not otherwise; that is to say

$V = 2$  is the only superficially divergent diagram

but

$V = 4$  is finite.

The three-pt function $E_B = 3$ diverges if $V = 1$ only – but there is no loop diagram in this case. It is finite.

$E_B \geqslant 4$ $\omega < 0$ and therefore the diagrams are superficial convergent.

We see that, in this theory, there is only a finite number (2) of divergent diagrams. It requires a $c\phi$ term in $\mathcal{L}$ to deal with the divergent tadpole —O and is then renormalizable

2) φ^4 $d_V = 0$ so

$$\omega = 4 - E_B$$

$E_B = 2, 4$ are superficially divergent

$E_B > 4$ " convergent

We will need to modify only the terms present in $\mathcal{L}$ to remove divergences.

3) $\varphi\bar{\psi}\psi$ $\quad d_v = 0$

$$\omega = 4 - E_B - \tfrac{3}{2}E_F$$

$E_F = 0 \quad E_B = 2, 3, 4$ superficially divergent

$E_F = 2 \quad E_B = 1$ " "

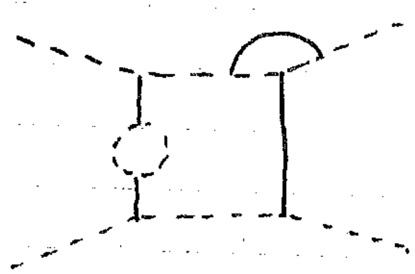
$E_F = 4 \quad \omega < 0$ " convergent

Here we see that we will need terms in the Lagrangian

$$\varphi^2,\ \varphi^3,\ \varphi^4,\ \varphi\bar{\psi}\psi$$

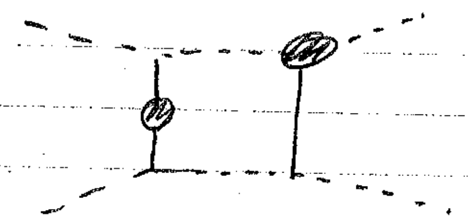
to be able to absorb all infinities.

The fact that a diagram is <u>superficially convergent</u> does not mean that it contains no divergences. For example

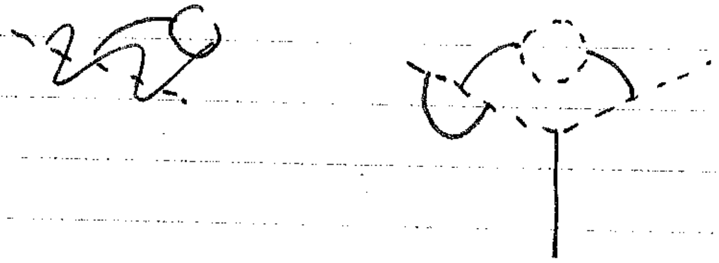


contains two divergent subintegrals. However these belong to proper diagrams of lower order - the two + three pt functions respectively. Once we have adjusted the parameters of

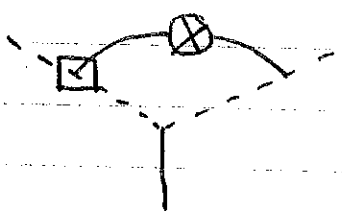
$\mathcal{L}$ to make these finite then the 4 pt function



is convergent. In general, once all proper subdiagrams have been rendered finite a superficially convergent diagram yields a finite result. On the other hand a superficially <u>divergent</u> diagram yields a new infinity. For example



again has two divergent subdiagrams. Once we have made the two + three point functions finite to order g^2, g^3 respectively we still have



$\otimes \sim g^2$
$\square \sim g^3$

which is a divergent contribution to the 3-pt function at order g^7.

The renormalization procedure works recursively in the way we have described; adjusting the parameter of $\mathcal{L}$ so as to give finite results at higher + higher orders in perturbation theory.

To prove that it works is highly non-trivial; one loop calculations that seem to work do not constitute a proof!!!

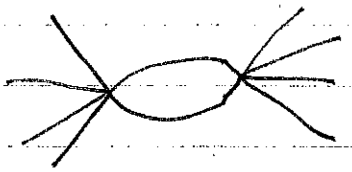
4) e.g. g^6 $\quad d_V = -2$ $\quad$ ie negative

$$\omega = 4 - E_B - \tfrac{3}{2} E_F + 2V$$

~~For diagrams of any fix~~

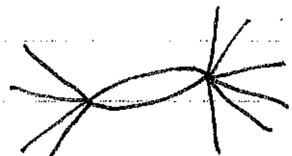
For Green's functions of any fixed $E_{B,F}$ there will be diagrams at sufficiently high order in p.t. (ie V) ~~such that~~ which are divergent. To absorb all infinities we would need an infinite number of terms in $\mathcal{L}$.

Let us just see explicitly what goes wrong for ϕ^6 theory.



divergent 8pt fn. so add as counterterm ϕ^8 to $\mathcal{L}$.

But now



divergent 10 pt fn. and so on. We land up needing an infinite number of coupling constants which destroys our original picture of physical quantities being related in a manner independent of the cut off — here we have to introduce a new arbitrary number for every distinct physical quantity!