QFT PS1 solutions

1 a)
Hamilton's equations are

$$\dot{q} = \{q, H\}, \quad \dot{p} = \{p, H\}.$$

A co-ordinate transformation

$$p = p' + P(p', q'), \quad q = q' + Q(p', q'),$$

is then canonical if

$$\{f, g\}_{pq} = \{f, g\}_{p'q'},$$

where

$$\{f, g\}_{pq} \equiv \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}.$$

The chain rule shows us that

$$\begin{aligned}\{f, g\}_{p'q'} &= \Big(\frac{\partial f}{\partial q}\frac{\partial q}{\partial q'} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial q'}\Big)\Big(\frac{\partial g}{\partial q}\frac{\partial q}{\partial p'} + \frac{\partial g}{\partial p}\frac{\partial p}{\partial p'}\Big) \\ &\quad - \Big(\frac{\partial f}{\partial q}\frac{\partial q}{\partial p'} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial p'}\Big)\Big(\frac{\partial g}{\partial q}\frac{\partial q}{\partial q'} + \frac{\partial g}{\partial p}\frac{\partial p}{\partial q'}\Big) \\ &= \{f, g\}_{pq} \cdot \{q, p\}_{q'p'} + \frac{\partial f}{\partial q}\frac{\partial g}{\partial q}\{q, q\}_{q'p'} + \frac{\partial f}{\partial p}\frac{\partial g}{\partial p}\{p, p\}_{q'p'}.\end{aligned}$$

Canonical $\iff \{q, p\}_{q'p'} = 1, \quad \{q, q\}_{q'p'} = \{p, p\}_{q'p'} = 0.$
Demanding the first condition holds gives us the result:

$$\begin{aligned}\{q, p\}_{q'p'} &= \Big(1 + \frac{\partial Q}{\partial q'}\Big)\Big(1 + \frac{\partial P}{\partial p'}\Big) - \frac{\partial Q}{\partial p'}\frac{\partial P}{\partial q'} \\ &= 1 + \frac{\partial Q}{\partial q'} + \frac{\partial P}{\partial p'} + \frac{\partial Q}{\partial q'}\frac{\partial P}{\partial p'} - \frac{\partial P}{\partial q'}\frac{\partial Q}{\partial q'} \\ &\stackrel{!}{=} 1\end{aligned}$$

b)
Consider the Jacobian

$$\begin{aligned}J &= \begin{pmatrix} \frac{\partial q}{\partial q'} & \frac{\partial p}{\partial q'} \\ \frac{\partial q}{\partial p'} & \frac{\partial p}{\partial p'} \end{pmatrix} \\ &= \begin{pmatrix} 1 + \frac{\partial Q}{\partial q'} & \frac{\partial P}{\partial q'} \\ \frac{\partial Q}{\partial p'} & 1 + \frac{\partial P}{\partial p'} \end{pmatrix}.\end{aligned}$$

Under the corresponding transformation,

$$dp\, dq \to dp' dq' = \det J \cdot dp\, dq.$$

The canonical condition above implies $\det J = 1$, so the phase space element is invariant.

c) For $p = p'$, $q = q'$, we have $P = Q = 0$. So infinitesimally we have $P = \epsilon\alpha(q', p')$, $Q = \epsilon\beta(q', p')$ with no $\mathcal{O}(\epsilon^0)$ piece. Plugging this into the canonical condition (a) gives us the result.

2 a)

$$\int_{-\infty}^{\infty} dx\, f(x)\,\delta(kx) \overset{y=kx}{=} \begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{dy}{k} f(y/k)\,\delta(y), & k > 0 \\ \displaystyle\int_{\infty}^{-\infty} \frac{dy}{k} f(y/k)\,\delta(y), & k < 0 \end{cases}$$
$$= \int_{-\infty}^{\infty} \frac{dy}{|k|} f(y/k)\,\delta(y)$$
$$= \frac{f(0)}{|k|} = \int_{-\infty}^{\infty} dx\, f(x)\,\frac{\delta(x)}{|k|}$$

b)

$$\int_{-\infty}^{\infty} dx\, f(x)\,\delta(g(x)) = \int_{x_0-\epsilon}^{x_0+\epsilon} dx\, f(x)\,\delta(g(x))$$
$$= \int_{x_0-\epsilon}^{x_0+\epsilon} dx\, f(x)\,\delta\big(g'(x_0)(x-x_0) + \mathcal{O}(\epsilon^2)\big)$$
$$= \int_{-\infty}^{\infty} dx\, f(x)\,\delta\big(g'(x_0)(x-x_0)\big)$$
$$= \int_{-\infty}^{\infty} dx\, f(x)\,\frac{\delta(x-x_0)}{|g'(x_0)|}$$

assuming there is only one zero $g(x_0) = 0$. We Taylor expand in the second line, and the last line follows from a).

c)

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dx\, f(x)\, K\frac{\epsilon}{x^2+\epsilon^2} = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dx\, f(x)\, K\frac{\epsilon}{(x+i\epsilon)(x-i\epsilon)}$$
$$= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dx\, f(x)\,\frac{iK}{2}[\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon}]$$

Now we need to do a contour integral. Let's assume $f(z)$ is well-behaved and tends to zero in the upper-half of the complex plane, so that we can close our contour there. The calculation for the other choice is the same (with a minus sign).

$$\int_{-\infty}^{\infty} dx\, f(x)\,[\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon}] = -\oint dz\,\frac{f(z)}{z-i\epsilon}$$
$$= -2\pi i f(0)$$
$$\implies K = \frac{1}{\pi}$$

Alternatively, the answer can be found with a change of variable $x = \epsilon\tan u$.

d)

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dx\, f(x)\,\hat{K}\frac{e^{-x^2/\epsilon^2}}{\epsilon} \overset{y=x/\epsilon}{=} \hat{K}\int_{-\infty}^{\infty} dy\, \lim_{\epsilon\to 0} f(\epsilon y)\, e^{-y^2}$$
$$= \hat{K}\int_{-\infty}^{\infty} dy\, f(0)\, e^{-y^2}$$
$$= \hat{K}\sqrt{\pi}\, f(0)$$
$$\implies \hat{K} = \frac{1}{\sqrt{\pi}}$$

3

$$[M^{\rho\sigma}]^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu$$

are the generators of the vector rep of the Lorentz group. For a boost along the z-axis of rapidity w ($\gamma \equiv \cosh w$), we need

$$\begin{aligned}
\Lambda^\mu{}_\nu &= \exp\Big(w[M^{03}]^\mu{}_\nu\Big) \\
&= \exp\left[w\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}\right] \\
&= P\exp\left[\begin{pmatrix}-w&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&w\end{pmatrix}\right]P^{-1} \\
&= \begin{pmatrix}\cosh w&0&0&\sinh w\\0&1&0&0\\0&0&1&0\\\sinh w&0&0&\cosh w\end{pmatrix},
\end{aligned}$$

where M^{03} is diagonalised by

$$P = \begin{pmatrix}-1&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&1\end{pmatrix}.$$

The Jacobian J associated with

$$p^\mu \to p'^\mu = \Lambda^\mu{}_\nu\, p^\nu$$

gives

$$\begin{aligned}
J &= \left|\frac{\partial p'^\mu}{\partial p^\nu}\right| \\
&= |\Lambda^\mu{}_\nu| \\
&= \cosh^2 w - \sinh^2 w = 1,
\end{aligned}$$

so the volume elemnt is Lorentz invariant.

For a timelike/null particle, $F(p) = \delta(p^2 - m^2)\theta(p^0)$ is Lorentz invariant. Hence

$$\begin{aligned}
\int d^4pF(p) &= \int dp^0\, d^3\mathbf{p}\,\delta\Big((p^0)^2 - (m^2 + \mathbf{p}^2)\Big)\theta(p^0) \\
&= \int \frac{d^3\mathbf{p}}{2\sqrt{m^2 + \mathbf{p}^2}}
\end{aligned}$$

is also invariant by 2(b).

4

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}m\omega_0^2(q_1^2 + q_2^2).$$

a)

$$\frac{\partial L}{\partial q_i} = -m\omega_0^2 q_i, \quad \frac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i$$
$$\implies \ddot{q}_i = -\omega_0^2 q_i.$$

b)

$$L = m\dot{z}\dot{z}^* - m\omega_0^2 zz^*$$

$$\frac{\delta L}{\delta z^*} = 0 \implies \ddot{z} = -\omega^2 z.$$

c)

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}^*,$$

so using $H + L = p_z\dot{z} + p_{z^*}\dot{z}^*$, we find

$$H = \frac{p_z p_{z*}}{2m} + m\omega_0^2 zz^*$$

d)

$$H = \frac{m}{4}(p_1^2 + p_2^2) + \frac{m\omega_0^2}{2}(q_1^2 + q_2^2),$$

and we have as usual

$$\begin{aligned} a_i &= \sqrt{\frac{m\omega_0}{2}}(q_i + \frac{i}{m\omega_0}p_i) \\ \implies H &= \omega_0(a_i^\dagger a_i + 1) \\ &= \omega_0(A^\dagger A + B^\dagger B + 1), \\ \text{where} \quad A &= \frac{1}{\sqrt{2}}(a_1 + ia_2), \quad B = \frac{1}{\sqrt{2}}(a_1 - ia_2) \end{aligned}$$

e) $[a_i, a_j^\dagger] = \delta_{ij}$, followed by algebra.

5. a) The Dirac equation

$$(i\partial\!\!\!/ - m)\psi = 0,$$

has plane wave solutions $\psi \propto \exp[-i(Et - \mathbf{p}\cdot\mathbf{x})]$ satisfying

$$d\psi_0 \equiv (\mathbf{p}\cdot\boldsymbol{\alpha} + m\beta - E)\psi_0 = 0$$

with

$$\beta = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}.$$

We therefore need the determinant condition

$$\det d = 0$$

for a non-invertible d to give non-trivial solutions.

We use that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$$

if $[C, D] = 0$ to find

$$\begin{aligned} \det d &= \det[-(\boldsymbol{\sigma}\cdot\mathbf{p} + E)(\boldsymbol{\sigma}\cdot\mathbf{p} - E) - m^2] \\ &= \det[E^2 - m^2 - p_i p_j \sigma_i \sigma_j] \\ &= \det[E^2 - m^2 - \frac{p_i p_j}{2}(\{\sigma_i, \sigma_j\} + [\sigma_i, \sigma_j])] \\ &= \det[E^2 - m^2 - \mathbf{p}\cdot\mathbf{p}] \\ &= (E^2 - m^2 - \mathbf{p}\cdot\mathbf{p})^2, \end{aligned}$$

where in the third line we have split the σ product into symmetric and antisymmetric parts, and the next line follows as $p_i p_j$ is totally symmetric and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$. Using the determinant condition gives the result.

b) The helicity operator is $h = \frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|} = \sigma^3$. Its two-component eigenspinors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with eigenvalues ± 1, respectively.