QFT Problem Set 2 Solutions

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Problem 1.

$$S = \int_0^T \mathrm{d}t \ \sum_k a^3 \Big[\frac{1}{2}\big(\partial_t \phi_k\big)^2 - \frac{1}{2}\kappa \sum_{\hat{\mu}} \Big(\phi_{k+\hat{\mu}} - \phi_k\Big)^2 - \frac{1}{2}\omega^2 \phi_k^2\Big] \tag{1}$$

(a) The Euler–Lagrange equations read,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}_k} &= a^3 \partial_t^2 \phi_k \\ \frac{\partial L}{\partial \phi_k} &= a^3 (2\kappa) \sum_{\hat{\mu}} \Big(\phi_{k+\hat{\mu}} - \phi_k\Big) - a^3 \omega^2 \phi_k \end{aligned} \tag{2}$$

The additional factor of 2 arises from the double counting in the summation of the edges. Setting $\kappa = \frac{1}{2}a^{-2}$, consider the contribution only along the $\hat{x}$ direction,

$$\begin{aligned} a^3 \frac{1}{a^2}\Big(\phi_{k+\hat{x}} - 2\phi_k + \phi_{k-\hat{x}}\Big) &= a^3 \frac{1}{a}\Big(\frac{\phi_{k+\hat{x}} - \phi_k}{a} - \frac{\phi_k - \phi_{k+\hat{x}}}{a}\Big) \\ &= a^3 \frac{1}{a}\Big(\tilde{\nabla}_{\hat{x}}\phi_k - \tilde{\nabla}_{\hat{x}}\phi_{k-\hat{x}}\Big) \\ &= a^3 \tilde{\nabla}_{\hat{x}}\tilde{\nabla}_{\hat{x}}\phi_{k-\hat{x}} \end{aligned} \tag{3}$$

And thus,

$$\partial_t^2 \phi_k - \tilde{\nabla}_{\hat{x}}\tilde{\nabla}_{\hat{x}}\phi_{k-\hat{x}} - \tilde{\nabla}_{\hat{y}}\tilde{\nabla}_{\hat{y}}\phi_{k-\hat{y}} - \tilde{\nabla}_{\hat{z}}\tilde{\nabla}_{\hat{z}}\phi_{k-\hat{z}} = -\omega^2 \phi_k \tag{4}$$

(b) Taking the limit $a \longrightarrow 0$,

$$\partial_t^2 \phi - \boldsymbol{\nabla}^2 \phi = -\omega^2 \phi \tag{5}$$

Problem 2.

$$\mathcal{L} = \frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}\omega^2\phi^2 \tag{6}$$

(a) Performing the variation,

$$\begin{aligned}
\delta S &= \int_0^T \mathrm{d}t \int_{V_\Sigma} \mathrm{d}^3\boldsymbol{x}\,\Big[\partial_t\phi_{\rm cl}\,\partial_t(\delta\phi) - \boldsymbol{\nabla}\phi_{\rm cl}\cdot\boldsymbol{\nabla}(\delta\phi) - \omega^2\phi_{\rm cl}\,(\delta\phi)\Big] \\
&= \int_{V_\Sigma} \mathrm{d}^3\boldsymbol{x}\,\left[\int_0^T \mathrm{d}t\,\Big(-(\delta\phi)\,\partial_t^2\phi_{\rm cl} - \boldsymbol{\nabla}\phi_{\rm cl}\cdot\boldsymbol{\nabla}(\delta\phi) - \omega^2\phi_{\rm cl}\,(\delta\phi)\Big) + \partial_t\phi_{\rm cl}\,\delta\phi\Big|_0^T\right] \\
&= \int_0^T \mathrm{d}t \int_{V_\Sigma} \mathrm{d}^3\boldsymbol{x}\,\Big(-(\delta\phi)\,\partial_t^2\phi_{\rm cl} - \boldsymbol{\nabla}\phi_{\rm cl}\cdot\boldsymbol{\nabla}(\delta\phi) - \omega^2\phi_{\rm cl}\,(\delta\phi)\Big) \\
&= \int_0^T \mathrm{d}t\,\left[\int_{V_\Sigma} \mathrm{d}^3\boldsymbol{x}\,(\delta\phi)\Big(-\partial_t^2\phi_{\rm cl} + \boldsymbol{\nabla}^2\phi_{\rm cl} - \omega^2\phi_{\rm cl}\Big) - \int_\Sigma \mathrm{d}A\,(\delta\phi)\,\boldsymbol{\nabla}\phi_{\rm cl}\cdot\hat{\boldsymbol{n}}\right]
\end{aligned} \tag{7}$$

We've integrated by parts and used Stokes' theorem in the second and fourth lines. We have used the fact that $\delta\phi(0,\mathbf{x}) = \delta\phi(T,\mathbf{x}) = 0$ in the third line.

(b) Let the surface Σ be a large sphere of radius R, and for simplicity assume that ϕ is spherically symmetric. Then,

$$\int_{S_R^2} \mathrm{d}A\,(\delta\phi)\,\boldsymbol{\nabla}\phi_{\rm cl}\cdot\hat{\boldsymbol{n}} = 2\pi R^2\,(\delta\phi)\,\hat{r}\cdot\boldsymbol{\nabla}\phi_{\rm cl} = 2\pi R^2\,(\delta\phi)\,\partial_r\phi_{\rm cl}(R) \tag{8}$$

For this to vanish, it is clear that $\phi_{\rm cl}(r)$ has to fall off *at least* as fast as

$$\begin{aligned}
\phi_{\rm cl}(r) \sim \frac{1}{r^{1+\epsilon}} \quad &\Longrightarrow \quad \partial_r\phi_{\rm cl}(r) \sim \frac{1}{r^{2+\epsilon}} \\
&\Longrightarrow \quad \lim_{R\to\infty} R^2\,\partial_r\phi_{\rm cl}(R) \sim \lim_{R\to\infty}\frac{1}{R^\epsilon} = 0
\end{aligned} \tag{9}$$

(c) The canonical momentum is,

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \partial_t\phi \tag{10}$$

And therefore, the Hamiltonian density is,

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}\omega^2\phi^2 \tag{11}$$

(d) The Hamiltonian is given by,

$$H = \int \mathrm{d}^3\boldsymbol{x}\,\mathcal{H} = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\,\Big(\pi^2 + (\boldsymbol{\nabla}\phi)^2 + \omega^2\phi^2\Big) \tag{12}$$

And therefore Hamilton's equations read,

$$\begin{aligned}
\dot{\phi}(t,\boldsymbol{x}) &= \frac{\delta H}{\delta\pi(t,\boldsymbol{x})} = \pi(t,\boldsymbol{x}) \\
\dot{\pi}(t,\boldsymbol{x}) &= -\frac{\delta H}{\delta\phi(t,\boldsymbol{x})} \\
&= -\int \mathrm{d}^3\boldsymbol{y}\,\Big(\boldsymbol{\nabla}\phi(t,\boldsymbol{y})\cdot\boldsymbol{\nabla}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) + \omega^2\phi(t,\boldsymbol{y})\ \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\Big) \\
&= \int \mathrm{d}^3\boldsymbol{y}\ \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\Big(\boldsymbol{\nabla}^2\phi(\boldsymbol{y}) - \omega^2\phi(\boldsymbol{y})\Big) \\
&= \boldsymbol{\nabla}^2\phi(t,\boldsymbol{x}) - \omega^2\phi(t,\boldsymbol{x})
\end{aligned} \tag{13}$$

Putting the two together,

$$\partial_t^2\phi(x) - \boldsymbol{\nabla}^2\phi(x) = -\omega^2\phi(x) \tag{14}$$

Problem 3.

$$\phi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a^\dagger_{-\boldsymbol{p}} + a_{\boldsymbol{p}}\right) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \quad \Longrightarrow \quad \tilde{\phi}(\boldsymbol{k}) = \frac{1}{\sqrt{2E_{\boldsymbol{k}}}} \left(a^\dagger_{-\boldsymbol{k}} + a_{\boldsymbol{k}}\right)$$

$$\pi(\boldsymbol{x}) = \mathrm{i} \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \sqrt{\frac{E_{\boldsymbol{p}}}{2}} \left(a^\dagger_{-\boldsymbol{p}} - a_{\boldsymbol{p}}\right) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \quad \Longrightarrow \quad \tilde{\pi}(\boldsymbol{k}) = \mathrm{i}\sqrt{\frac{E_{\boldsymbol{k}}}{2}} \left(a^\dagger_{-\boldsymbol{k}} - a_{\boldsymbol{k}}\right)$$

$$\Longrightarrow \; a^\dagger_{\boldsymbol{k}} = \frac{1}{\sqrt{2E_{\boldsymbol{k}}}} \left(E_{\boldsymbol{k}}\, \tilde{\phi}(-\boldsymbol{k}) - \mathrm{i}\, \tilde{\pi}(-\boldsymbol{k})\right) \;,\quad a_{\boldsymbol{k}} = \frac{1}{\sqrt{2E_{\boldsymbol{k}}}} \left(E_{\boldsymbol{k}}\, \tilde{\phi}(\boldsymbol{k}) + \mathrm{i}\, \tilde{\pi}(\boldsymbol{k})\right) \tag{15}$$

Using this, consider,

$$\begin{aligned} a^\dagger_{\boldsymbol{k}}\, a_{\boldsymbol{k}} &= \frac{1}{2E_{\boldsymbol{k}}} \left[E^2_{\boldsymbol{k}}\, |\tilde{\phi}(\boldsymbol{k})|^2 + |\tilde{\pi}(\boldsymbol{k})|^2 + \mathrm{i}E_{\boldsymbol{k}}\, \tilde{\phi}(-\boldsymbol{k})\, \tilde{\pi}(\boldsymbol{k}) - \mathrm{i}E_{\boldsymbol{k}}\, \tilde{\pi}(-\boldsymbol{k})\, \tilde{\phi}(\boldsymbol{k})\right] \\ a_{-\boldsymbol{k}}\, a^\dagger_{-\boldsymbol{k}} &= \frac{1}{2E_{\boldsymbol{k}}} \left[E^2_{\boldsymbol{k}}\, |\tilde{\phi}(\boldsymbol{k})|^2 + |\tilde{\pi}(\boldsymbol{k})|^2 - \mathrm{i}E_{\boldsymbol{k}}\, \tilde{\phi}(-\boldsymbol{k})\, \tilde{\pi}(\boldsymbol{k}) + \mathrm{i}E_{\boldsymbol{k}}\, \tilde{\pi}(-\boldsymbol{k})\, \tilde{\phi}(\boldsymbol{k})\right] \end{aligned} \tag{16}$$

In the above equations,

$$|\tilde{\phi}(\boldsymbol{k})|^2 = \tilde{\phi}(\boldsymbol{k})\, \tilde{\phi}(-\boldsymbol{k}) \quad , \quad |\tilde{\pi}(\boldsymbol{k})|^2 = \tilde{\pi}(\boldsymbol{k})\, \tilde{\pi}(-\boldsymbol{k}) \tag{17}$$

(a)

$$\begin{aligned} H &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\; E_{\boldsymbol{p}}\, a^\dagger_{\boldsymbol{p}}\, a_{\boldsymbol{p}} = :\frac{1}{2} \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\; E_{\boldsymbol{p}} \left(a^\dagger_{\boldsymbol{p}}\, a_{\boldsymbol{p}} + a_{-\boldsymbol{p}}\, a^\dagger_{-\boldsymbol{p}}\right) : \\ &= :\frac{1}{2} \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \left(E^2_{\boldsymbol{k}}\, |\tilde{\phi}(\boldsymbol{k})|^2 + |\tilde{\pi}(\boldsymbol{k})|^2\right) : \\ &= :\frac{1}{2} \int \mathrm{d}^3\boldsymbol{x} \left(\pi^2(\boldsymbol{x}) + (\boldsymbol{\nabla}\phi)^2(\boldsymbol{x}) + \omega^2\phi^2(\boldsymbol{x})\right) : \end{aligned} \tag{18}$$

(b)

$$\begin{aligned} \boldsymbol{P} &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\; \boldsymbol{p}\, a^\dagger_{\boldsymbol{p}}\, a_{\boldsymbol{p}} = \frac{1}{2} : \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \left(\boldsymbol{p}\, a^\dagger_{\boldsymbol{p}}\, a_{\boldsymbol{p}} - \boldsymbol{p}\, a^\dagger_{-\boldsymbol{p}}\, a_{-\boldsymbol{p}}\right) : \\ &= :\frac{1}{2} \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \left(\mathrm{i}\boldsymbol{p}\, \tilde{\phi}(-\boldsymbol{p})\, \tilde{\pi}(\boldsymbol{p}) - \mathrm{i}\boldsymbol{p}\, \tilde{\pi}(-\boldsymbol{p})\, \tilde{\phi}(\boldsymbol{p})\right) : \\ &= \; - : \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\; (\mathrm{i}\boldsymbol{p})\, \tilde{\pi}(-\boldsymbol{p})\, \tilde{\phi}(\boldsymbol{p}) := - : \int \mathrm{d}^3\boldsymbol{x}\; \pi(\boldsymbol{x})\, \boldsymbol{\nabla}\phi(\boldsymbol{x}) : \end{aligned} \tag{19}$$

(c) Note,

$$a^\dagger_{\boldsymbol{k}}\, a_{\boldsymbol{k}}\, a^\dagger_{\boldsymbol{p}}\, a^\dagger_{\boldsymbol{p}'}\, |0\rangle = (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p})\; a^\dagger_{\boldsymbol{k}}\, a^\dagger_{\boldsymbol{p}'}\, |0\rangle + (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p}')\; a^\dagger_{\boldsymbol{k}}\, a^\dagger_{\boldsymbol{p}}\, |0\rangle \tag{20}$$

And therefore,

$$
\begin{aligned}
H\,|\boldsymbol{p},\boldsymbol{p'}\rangle &= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\; E_{\boldsymbol{k}}\, a^\dagger_{\boldsymbol{k}}\, a_{\boldsymbol{k}}\, a^\dagger_{\boldsymbol{p}}\, a^\dagger_{\boldsymbol{p'}}\,|0\rangle \\
&= E_{\boldsymbol{p}}\, a^\dagger_{\boldsymbol{p}}\, a^\dagger_{\boldsymbol{p'}}\,|0\rangle + E_{\boldsymbol{p'}}\, a^\dagger_{\boldsymbol{p}}\, a^\dagger_{\boldsymbol{p'}}\,|0\rangle = \big(E_{\boldsymbol{p}} + E_{\boldsymbol{p'}}\big)\,|\boldsymbol{p},\boldsymbol{p'}\rangle \\
\boldsymbol{P}\,|\boldsymbol{p},\boldsymbol{p'}\rangle &= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\; \boldsymbol{k}\, a^\dagger_{\boldsymbol{k}}\, a_{\boldsymbol{k}}\, a^\dagger_{\boldsymbol{p}}\, a^\dagger_{\boldsymbol{p'}}\,|0\rangle \\
&= \boldsymbol{p}\, a^\dagger_{\boldsymbol{p}}\, a^\dagger_{\boldsymbol{p'}}\,|0\rangle + \boldsymbol{p'}\, a^\dagger_{\boldsymbol{p}}\, a^\dagger_{\boldsymbol{p'}}\,|0\rangle = \big(\boldsymbol{p} + \boldsymbol{p'}\big)\,|\boldsymbol{p},\boldsymbol{p'}\rangle
\end{aligned}
\tag{21}
$$

(d)

$$
\begin{aligned}
\langle 0|\,\tilde{\phi}(\boldsymbol{k})\,\tilde{\phi}(\boldsymbol{k'})\,|\boldsymbol{p},\boldsymbol{p'}\rangle &= \frac{1}{2\sqrt{E_{\boldsymbol{k}}\,E_{\boldsymbol{k'}}}}\,\langle 0|\,a_{\boldsymbol{k}}\,a_{\boldsymbol{k'}}\,a^\dagger_{\boldsymbol{p}}\,a^\dagger_{\boldsymbol{p'}}\,|0\rangle \\
&= \frac{(2\pi)^6}{2\sqrt{E_{\boldsymbol{k}}\,E_{\boldsymbol{k'}}}}\Big(\delta^3(\boldsymbol{k}-\boldsymbol{p'})\;\delta^3(\boldsymbol{k'}-\boldsymbol{p}) + \delta^3(\boldsymbol{k}-\boldsymbol{p})\;\delta^3(\boldsymbol{k'}-\boldsymbol{p'})\Big)
\end{aligned}
\tag{22}
$$

And thus,

$$
\begin{aligned}
\langle 0|\,\phi(\boldsymbol{x})\,\phi(\boldsymbol{y})\,|\boldsymbol{p},\boldsymbol{p'}\rangle &= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\frac{\mathrm{d}^3\boldsymbol{k'}}{(2\pi)^3}\;\langle 0|\,\tilde{\phi}(\boldsymbol{k})\,\tilde{\phi}(\boldsymbol{k'})\,|\boldsymbol{p},\boldsymbol{p'}\rangle\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}+\mathrm{i}\boldsymbol{k'}\cdot\boldsymbol{y}} \\
&= \frac{1}{2\sqrt{E_{\boldsymbol{p}}\,E_{\boldsymbol{p'}}}}\Big(\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}+\mathrm{i}\boldsymbol{p'}\cdot\boldsymbol{y}} + \mathrm{e}^{\mathrm{i}\boldsymbol{p'}\cdot\boldsymbol{x}+\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{y}}\Big)
\end{aligned}
\tag{23}
$$

Problem 4.

$$\Delta(x) = [\phi(x), \phi(0)] = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(\mathrm{e}^{-\mathrm{i}p\cdot x} - \mathrm{e}^{\mathrm{i}p\cdot x}\right) \;\; ; \; E_p = \sqrt{\boldsymbol{p}^2 + m^2} \tag{24}$$

(a) Let $x = (t, \boldsymbol{x})$. Choosing spherical polar co-ordinates in *momentum* space, with the polar axis along the direction of $\boldsymbol{x}$, we have,

$$\begin{aligned}
\Delta(x) &= \frac{1}{(2\pi)^3} \int_0^\infty \mathrm{d}p\; p^2 \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}(\cos\theta)\; \frac{1}{2E_p} \left(\mathrm{e}^{-\mathrm{i}E_p t + \mathrm{i}p|\boldsymbol{x}|\cos\theta} - \mathrm{e}^{\mathrm{i}E_p t - \mathrm{i}p|\boldsymbol{x}|\cos\theta}\right) \\
&= \frac{-\mathrm{i}}{2(2\pi)^2|\boldsymbol{x}|} \int_0^\infty \mathrm{d}p\; \frac{p}{E_p} \left(\mathrm{e}^{-\mathrm{i}E_p t + \mathrm{i}p|\boldsymbol{x}|} - \mathrm{e}^{-\mathrm{i}E_p t - \mathrm{i}p|\boldsymbol{x}|} + \mathrm{e}^{\mathrm{i}E_p t - \mathrm{i}p|\boldsymbol{x}|} - \mathrm{e}^{\mathrm{i}E_p t + \mathrm{i}p|\boldsymbol{x}|}\right) \\
&= \frac{-\mathrm{i}}{4\pi^2|\boldsymbol{x}|} \int_0^\infty \mathrm{d}p\; \frac{p}{\sqrt{p^2+m^2}}\; \mathbf{Re}\left[\mathrm{e}^{\mathrm{i}E_p t - \mathrm{i}p|\boldsymbol{x}|} - \mathrm{e}^{\mathrm{i}E_p t + \mathrm{i}p|\boldsymbol{x}|}\right] \\
&= \frac{-\mathrm{i}}{8\pi^2|\boldsymbol{x}|} \int_{-\infty}^\infty \mathrm{d}p\; \frac{p}{\sqrt{p^2+m^2}}\; \mathbf{Re}\left[\mathrm{e}^{\mathrm{i}E_p t - \mathrm{i}p|\boldsymbol{x}|} - \mathrm{e}^{\mathrm{i}E_p t + \mathrm{i}p|\boldsymbol{x}|}\right]
\end{aligned} \tag{25}$$

Now, using rapidity parameters. $t = s\cosh\tau,\; |\boldsymbol{x}| = s\sinh\tau,\;\; p = m\sinh\phi \implies \mathrm{d}p = m\cosh\phi\; \mathrm{d}\phi,\;\; E_p = m\cosh\phi$. Further,

$$\begin{aligned}
E_p t - p|\boldsymbol{x}| &= ms\big(\cosh\phi\;\cosh\tau - \sinh\phi\;\sinh\tau\big) = ms\cosh(\phi - \tau) \\
E_p t + p|\boldsymbol{x}| &= ms\big(\cosh\phi\;\cosh\tau + \sinh\phi\;\sinh\tau\big) = ms\cosh(\phi + \tau)
\end{aligned} \tag{26}$$

Putting this all together,

$$\begin{aligned}
\Delta(x) &= \frac{-\mathrm{i}}{8\pi^2 s\sinh\tau} \int_{-\infty}^\infty \mathrm{d}\phi\; \frac{m^2\sinh\phi\;\cosh\phi}{m\cosh\phi}\; \mathbf{Re}\left[\mathrm{e}^{\mathrm{i}ms\cosh(\phi-\tau)} - \mathrm{e}^{\mathrm{i}ms\cosh(\phi+\tau)}\right] \\
&= \frac{-\mathrm{i}}{8\pi^2 s} \int_{-\infty}^\infty \mathrm{d}\phi\; \frac{m\sinh\phi}{\sinh\tau}\; \mathbf{Re}\left[\mathrm{e}^{\mathrm{i}ms\cosh(\phi-\tau)} - \mathrm{e}^{\mathrm{i}ms\cosh(\phi+\tau)}\right]
\end{aligned} \tag{27}$$

(b) Consider,

$$\begin{aligned}
\Delta^{(1)}(x) &= \frac{-\mathrm{i}}{8\pi^2 s} \int_{-\infty}^\infty \mathrm{d}\phi\; \frac{m\sinh\phi}{\sinh\tau}\; \mathbf{Re}\left[\mathrm{e}^{\mathrm{i}ms\cosh(\phi-\tau)}\right] \\
&= \frac{-\mathrm{i}}{8\pi^2 s} \int_{-\infty}^\infty \mathrm{d}\phi\; \frac{m\sinh(\phi+\tau)}{\sinh\tau}\; \mathbf{Re}\left[\mathrm{e}^{\mathrm{i}ms\cosh\phi}\right] \\
&= \frac{-\mathrm{i}}{8\pi^2 s} \int_{-\infty}^\infty \mathrm{d}\phi\; \frac{m\big(\sinh\phi\;\cosh\tau + \cosh\phi\;\sinh\tau\big)}{\sinh\tau}\; \mathbf{Re}\left[\mathrm{e}^{\mathrm{i}ms\cosh\phi}\right] \\
&= \frac{-\mathrm{i}}{8\pi^2 s} \int_{-\infty}^\infty \mathrm{d}\phi\; \frac{m\cosh\phi\;\sinh\tau}{\sinh\tau}\; \mathbf{Re}\left[\mathrm{e}^{\mathrm{i}ms\cosh\phi}\right] \\
&= \frac{-\mathrm{i}m}{8\pi^2 s} \int_{-\infty}^\infty \mathrm{d}\phi\; \cosh\phi\; \mathbf{Re}\left[\mathrm{e}^{\mathrm{i}ms\cosh\phi}\right]
\end{aligned} \tag{28}$$

And similarly,

$$\begin{aligned}\Delta^{(2)}(x) &= \frac{-\mathrm{i}}{8\pi^2 s}\int_{-\infty}^{\infty}\mathrm{d}\phi\ \frac{m\sinh\phi}{\sinh\tau}\ \mathbf{Re}\Big[-\mathrm{e}^{ims\cosh(\phi+\tau)}\Big] \\ &= \frac{-\mathrm{i}m}{8\pi^2 s}\int_{-\infty}^{\infty}\mathrm{d}\phi\ \cosh\phi\ \mathbf{Re}\Big[\mathrm{e}^{ims\cosh\phi}\Big]\end{aligned} \tag{29}$$

And thus,

$$\Delta(x) = \frac{-\mathrm{i}m}{4\pi^2 s}\int_{-\infty}^{\infty}\mathrm{d}\phi\ \cosh\phi\ \mathbf{Re}\Big[\mathrm{e}^{ims\cosh\phi}\Big] \tag{30}$$

This representation expresses the unequal time propagator purely in terms of the spacetime interval s and mass m.

(c) The Mehler Sonine representation of the Bessel function of the first kind is,

$$J_\nu(z) = \frac{2}{\pi}\int_0^{\infty}\mathrm{d}t\ \cosh(\nu t)\ \sin\Big(z\cosh t - \frac{\pi}{2}\nu\, t\Big)\quad ,\quad |\mathbf{Re}\ \nu| < 1 \tag{31}$$

$$J_1(ms) = \frac{2}{\pi}\int_0^{\infty}\mathrm{d}t\ \cosh t\ \cos\big(ms\cosh t\big) = \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}t\ \cosh t\ \cos\big(ms\cosh t\big)$$

$$\implies \Delta(x) = \frac{-\mathrm{i}m}{4\pi s}J_1(ms) \tag{32}$$

Note: This can also be expressed as the real part of a Hankel function (ref. Lebedev – Special Functions and their Applications, eq. 5.10.14),

$$H_\nu^{(1)}(z) = \frac{\mathrm{e}^{-\frac{\mathrm{i}\pi\nu}{2}}}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}u\ \mathrm{e}^{\mathrm{i}z\cosh u - \nu u}$$

The two formulae can easily shown to be equal.

There are two subtleties here.

1. The Mehler Sonine representation of the Bessel function is strictly only valid for $|\mathbf{Re}\,\nu| < 1$ – we have used the boundary value of $\nu = 1$.

2. In going over to the rapidity parameters, we set $t = s\cosh\tau$, $|\boldsymbol{x}| = s\sinh\tau$.

$$\implies t^2 - |\boldsymbol{x}|^2 = s^2\big(\cosh^2\tau - \sinh^2\tau\big) = s^2 > 0 \tag{33}$$

   That is, we are only considering *timelike* intervals.