INTRODUCTION TO QFT 2019: PROBLEM SET 3 SOLUTIONS

1. Particle Projection Operators

We will be interested in the product

$$\langle \mathbf{p}_1, \ldots, \mathbf{p}_n | \, \mathbf{k}_n, \ldots, \mathbf{k}_1 \rangle = \langle 0 | \prod_{i=1}^{n} \sqrt{2E_{\mathbf{p}_i}} a_{\mathbf{p}_i} \prod_{j=1}^{n} \sqrt{2E_{\mathbf{k}_j}} a^\dagger_{\mathbf{k}_j} |0\rangle \, . \tag{1}$$

First we prove the following identity:

$$\prod_{i=1}^{m} a_{\mathbf{p}_i} \prod_{j=1}^{n} a^\dagger_{\mathbf{k}_j} |0\rangle = (2\pi)^{3m} \sum_{i_1=1}^{n} \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^{n} \cdots \sum_{\substack{i_m=1 \\ i_m \neq i_1, \ldots, i_{m-1}}}^{n} \delta^3(\mathbf{p}_1 - \mathbf{k}_{i_1}) \cdots \delta^3(\mathbf{p}_m - \mathbf{k}_{i_m}) \prod_{\substack{j=1 \\ j \neq i_1, \ldots, i_m}}^{n} a^\dagger_{\mathbf{k}_j} |0\rangle \, , \tag{2}$$

with $m < n$. Let us proceed by induction on m. For the sake of readability we drop $|0\rangle$ in our derivation, but all operators act on the vacuum $|0\rangle$. For $m = 1$ one has

$$\begin{aligned} a_{\mathbf{p}_1} \prod_{j=1}^{n} a^\dagger_{\mathbf{k}_j} &= (2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{k}_n) \prod_{\substack{i=1 \\ i \neq n}}^{n} a^\dagger_{\mathbf{k}_j} + a^\dagger_{\mathbf{k}_n} a_{\mathbf{p}_1} \prod_{j=1}^{n-1} a^\dagger_{\mathbf{k}_j} \\ &= (2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{k}_n) \prod_{\substack{j=1 \\ j \neq n}}^{n} a^\dagger_{\mathbf{k}_j} + (2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{k}_{n-1}) \prod_{\substack{j=1 \\ j \neq n-1}}^{n} a^\dagger_{\mathbf{k}_j} + a^\dagger_{\mathbf{k}_n} a^\dagger_{\mathbf{k}_{n-1}} a_{\mathbf{p}_1} \prod_{j=1}^{n-2} a^\dagger_{\mathbf{k}_j} \\ &= \ldots = \sum_{i_1=1}^{n} (2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{k}_{i_1}) \prod_{\substack{j=1 \\ j \neq i_1}}^{n} a^\dagger_{\mathbf{k}_j} \, . \end{aligned} \tag{3}$$

Take now $m < n$ and rewrite the product on the lhs of (2) as

$$\prod_{\ell=1}^{m} a_{\mathbf{p}_\ell} \prod_{j=1}^{n} a^\dagger_{\mathbf{k}_j} = a_{\mathbf{p}_m} \prod_{\ell=1}^{m-1} a_{\mathbf{p}_\ell} \prod_{j=1}^{n} a^\dagger_{\mathbf{k}_j} \tag{4}$$

and assume (2) to hold for $m - 1$, thus

$$\prod_{\ell=1}^{m} a_{\mathbf{p}_\ell} \prod_{j=1}^{n} a^\dagger_{\mathbf{k}_j} = a_{\mathbf{p}_m} (2\pi)^{3(m-1)} \left(\sum_{i_1=1}^{n} \cdots \sum_{\substack{i_{m-1}=1 \\ i_{m-1} \neq i_1, \ldots, i_{m-2}}}^{n} \delta^3(\mathbf{p}_1 - \mathbf{k}_{i_1}) \cdots \delta^3(\mathbf{p}_{m-1} - \mathbf{k}_{i_{m-1}}) \prod_{\substack{j=1 \\ j \neq i_1, \ldots, i_{m-1}}}^{n} a^\dagger_{\mathbf{k}_j} \right) . \tag{5}$$

Using (3), the product of $a_{\mathbf{p}_m}$ with the $a^\dagger_{\mathbf{k}_j}$ reads

$$a_{\mathbf{p}_m} \prod_{\substack{j=1 \\ j \neq i_1, \ldots, i_{m-1}}}^{n} a^\dagger_{\mathbf{k}_j} = \sum_{\substack{i_m=1 \\ i_m \neq i_1, \ldots, i_{m-1}}} (2\pi)^3 \delta^3(\mathbf{p}_m - \mathbf{k}_{i_m}) \prod_{\substack{j=1 \\ j \neq i_1, \ldots, i_m}}^{n} a^\dagger_{\mathbf{k}_j} \, . \tag{6}$$

Plugging the above in (5) we finally get

$$\prod_{i=1}^{m} a_{\mathbf{p}_i} \prod_{j=1}^{n} a^\dagger_{\mathbf{k}_j} = (2\pi)^{3m} \sum_{i_1=1}^{n} \cdots \sum_{\substack{i_m=1 \\ i_m \neq i_1, \ldots, i_{m-1}}}^{n} \delta^3(\mathbf{p}_1 - \mathbf{k}_{i_1}) \cdots \delta^3(\mathbf{p}_m - \mathbf{k}_{i_m}) \prod_{\substack{j=1 \\ j \neq i_1, \ldots, i_m}}^{n} a^\dagger_{\mathbf{k}_j} \, . \tag{7}$$

If we select $m = n$, the product of $a^\dagger_{\mathbf{k}_j}$ operators on the rhs of (2) reduces to 1 and rest is simply the sum over all permutations of n-indices, $\mathcal{S}_n$. Therefore, for real scalar fields we get

$$\langle \mathbf{p}_1, \ldots, \mathbf{p}_n | \, \mathbf{k}_n, \ldots, \mathbf{k}_1 \rangle = (2\pi)^{3n} \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} 2E_{\mathbf{p}_i} \delta^3(\mathbf{p}_i - \mathbf{k}_{\sigma(i)}) \,. \tag{8}$$

$$\begin{aligned}
\langle \mathbf{q}_1, \ldots, \mathbf{q}_n | \, P_n \, | \mathbf{k}_n, \ldots, \mathbf{k}_1 \rangle &= \langle \mathbf{q}_1, \ldots, \mathbf{q}_n | \, \frac{1}{n!} \prod_{j=1}^{n} \int \frac{\mathrm{d}^3 \mathbf{p}_j}{(2\pi)^3 2E_{\mathbf{p}_j}} \, | \mathbf{p}_n, \ldots, \mathbf{p}_1 \rangle \, \langle \mathbf{p}_1, \ldots, \mathbf{p}_n | \, \mathbf{k}_n, \ldots, \mathbf{k}_1 \rangle \\
&= \frac{1}{n!} \prod_{j=1}^{n} \int \frac{\mathrm{d}^3 \mathbf{p}_j}{(2\pi)^3 2E_{\mathbf{p}_j}} \, \langle \mathbf{q}_1, \ldots, \mathbf{q}_n | \, \mathbf{p}_n, \ldots, \mathbf{p}_1 \rangle \, \langle \mathbf{p}_1, \ldots, \mathbf{p}_n | \, \mathbf{k}_n, \ldots, \mathbf{k}_1 \rangle \\
&= \frac{1}{n!} \prod_{j=1}^{n} \int \frac{\mathrm{d}^3 \mathbf{p}_j}{(2\pi)^3 2E_{\mathbf{p}_j}} \, \langle \mathbf{q}_1, \ldots, \mathbf{q}_n | \, \mathbf{p}_n, \ldots, \mathbf{p}_1 \rangle \left((2\pi)^{3n} \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} 2E_{\mathbf{p}_i} \delta^3(\mathbf{p}_i - \mathbf{k}_{\sigma(i)}) \right) \\
&= \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} \int \mathrm{d}^3 \mathbf{p}_i \, \delta^3(\mathbf{p}_i - \mathbf{k}_{\sigma(i)}) \, \langle \mathbf{q}_1, \ldots, \mathbf{q}_n | \, \mathbf{p}_n, \ldots, \mathbf{p}_1 \rangle \\
&= \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \langle \mathbf{q}_1, \ldots, \mathbf{q}_n | \, \mathbf{k}_{\sigma(n)}, \ldots, \mathbf{k}_{\sigma(1)} \rangle \,.
\end{aligned} \tag{9}$$

Permuting n bosonic real fields gives the same physical state. This is a consequence of the fact that creation, as well as annihilation operators commute, i.e.

$$[a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = [a_{\mathbf{p}}, a_{\mathbf{q}}] = 0 \,. \tag{10}$$

Since there are $n!$ permuations in $\mathcal{S}_n$ we finally get

$$\frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \langle \mathbf{q}_1, \ldots, \mathbf{q}_n | \, \mathbf{k}_{\sigma(n)}, \ldots, \mathbf{k}_{\sigma(1)} = \langle \mathbf{q}_1, \ldots, \mathbf{q}_n | \, \mathbf{k}_n, \ldots, \mathbf{k}_1 \rangle \,, \tag{11}$$

thus P_n acts as the identity operator.

2. The Quantized Dirac Field

a) Let us recall that

$$\left\{a_{\mathbf{p}}^{s}, a_{\mathbf{k}}^{r\,\dagger}\right\} = \left\{b_{\mathbf{p}}^{s}, b_{\mathbf{k}}^{r\,\dagger}\right\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\delta^{rs} \tag{12}$$

and all other anticommutators are zero.

$$[\psi, H] = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{E_{\mathbf{k}}}{\sqrt{2E_{\mathbf{p}}}}\sum_{r=\pm}\sum_{s=\pm}\left[e^{-ip\cdot x}\left(\left[a_{\mathbf{p}}^{s}, a_{\mathbf{k}}^{r\,\dagger}a_{\mathbf{k}}^{r}\right] + \left[a_{\mathbf{p}}^{s}, b_{\mathbf{k}}^{r\,\dagger}b_{\mathbf{k}}^{r}\right]\right)u^s(\mathbf{p})\right.$$
$$\left. + \quad e^{ip\cdot x}\left(\left[b_{\mathbf{p}}^{s\,\dagger}, a_{\mathbf{k}}^{r\,\dagger}a_{\mathbf{k}}^{r}\right] + \left[b_{\mathbf{p}}^{s\,\dagger}, b_{\mathbf{k}}^{r\,\dagger}b_{\mathbf{k}}^{r}\right]\right)v^s(\mathbf{p})\right]. \tag{13}$$

For the commutators in (13) we have

$$\begin{aligned}\left[a_{\mathbf{p}}^{s}, a_{\mathbf{k}}^{r\,\dagger}a_{\mathbf{k}}^{r}\right] &= a_{\mathbf{p}}^{s}a_{\mathbf{k}}^{r\,\dagger}a_{\mathbf{k}}^{r} - a_{\mathbf{k}}^{r\,\dagger}a_{\mathbf{k}}^{r}a_{\mathbf{p}}^{s} = a_{\mathbf{p}}^{s}a_{\mathbf{k}}^{r\,\dagger}a_{\mathbf{k}}^{r} + a_{\mathbf{k}}^{r\,\dagger}a_{\mathbf{p}}^{s}a_{\mathbf{k}}^{r} \\ &= a_{\mathbf{p}}^{s}a_{\mathbf{k}}^{r\,\dagger}a_{\mathbf{k}}^{r} - a_{\mathbf{p}}^{s}a_{\mathbf{k}}^{r\,\dagger}a_{\mathbf{k}}^{r} + \left\{a_{\mathbf{p}}^{s}, a_{\mathbf{k}}^{r\,\dagger}\right\}a_{\mathbf{k}}^{r} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\delta^{rs}a_{\mathbf{k}}^{r}\,.\end{aligned} \tag{14}$$

and equivalently

$$\begin{aligned}\left[b_{\mathbf{p}}^{s\,\dagger}, b_{\mathbf{k}}^{r\,\dagger}b_{\mathbf{k}}^{r}\right] &= b_{\mathbf{p}}^{s\,\dagger}b_{\mathbf{k}}^{r\,\dagger}b_{\mathbf{k}}^{r} - b_{\mathbf{k}}^{r\,\dagger}b_{\mathbf{k}}^{r}b_{\mathbf{p}}^{s\,\dagger} = -b_{\mathbf{k}}^{r\,\dagger}b_{\mathbf{p}}^{s\,\dagger}b_{\mathbf{k}}^{r} - b_{\mathbf{k}}^{r\,\dagger}b_{\mathbf{k}}^{r}b_{\mathbf{p}}^{s\,\dagger} \\ &= -b_{\mathbf{k}}^{r\,\dagger}\left\{b_{\mathbf{k}}^{r}, b_{\mathbf{p}}^{s\,\dagger}\right\} + b_{\mathbf{k}}^{r\,\dagger}b_{\mathbf{k}}^{r}b_{\mathbf{p}}^{s\,\dagger} - b_{\mathbf{k}}^{r\,\dagger}b_{\mathbf{k}}^{r}b_{\mathbf{p}}^{s\,\dagger} = -(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\delta^{rs}b_{\mathbf{k}}^{r\,\dagger}\,.\end{aligned} \tag{15}$$

Applying the same algebraic manipulations one can readily show that

$$\left[a_{\mathbf{p}}^{s}, b_{\mathbf{k}}^{r\,\dagger}b_{\mathbf{k}}^{r}\right] = \left[b_{\mathbf{p}}^{s\,\dagger}, a_{\mathbf{k}}^{r\,\dagger}a_{\mathbf{k}}^{r}\right] = 0\,. \tag{16}$$

Plugging (14) and (15) into (13) we get

$$\begin{aligned}[\psi, H] &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{E_{\mathbf{k}}}{\sqrt{2E_{\mathbf{p}}}}\sum_{r=\pm}\sum_{s=\pm}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\delta^{rs}\left[e^{-ip\cdot x}a_{\mathbf{k}}^{r}u^s(\mathbf{p}) - e^{ip\cdot x}b_{\mathbf{k}}^{r\,\dagger}v^s(\mathbf{p})\right] \\ &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{i}{\sqrt{2E_{\mathbf{p}}}}\sum_{s=\pm}\left[(-iE_{\mathbf{p}})e^{-ip\cdot x}a_{\mathbf{p}}^{s}u^s(\mathbf{p}) + (iE_{\mathbf{p}})e^{ip\cdot x}b_{\mathbf{p}}^{s\,\dagger}v^s(\mathbf{p})\right] \\ &= i\frac{\mathrm{d}}{\mathrm{d}\,t}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{i}{\sqrt{2E_{\mathbf{p}}}}\sum_{s=\pm}\left[e^{-ip\cdot x}a_{\mathbf{p}}^{s}u^s(\mathbf{p}) + e^{ip\cdot x}b_{\mathbf{p}}^{s\,\dagger}v^s(\mathbf{p})\right]\end{aligned} \tag{17}$$

therefore

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi = [\psi, H]\,. \tag{18}$$

b)

$$Ha_{\mathbf{p}}^{s\,\dagger}|\psi\rangle = a_{\mathbf{p}}^{s\,\dagger}H|\psi\rangle + \left[H, a_{\mathbf{p}}^{s\,\dagger}\right]|\psi\rangle = E_\psi a_{\mathbf{p}}^{s\,\dagger}|\psi\rangle + \left[H, a_{\mathbf{p}}^{s\,\dagger}\right]|\psi\rangle\,, \tag{19}$$

where we used the fact that $|\psi\rangle$ is an eigenstate of the Hamiltonian. As for the commutator we have

$$\left[H, a_{\mathbf{p}}^{s\,\dagger}\right] = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}E_{\mathbf{k}}\sum_{r=\pm}\left[a_{\mathbf{k}}^{r\,\dagger}a_{\mathbf{k}}^{r}, a_{\mathbf{p}}^{s\,\dagger}\right] = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}E_{\mathbf{k}}\sum_{r=\pm}(2\pi)^3\delta^{rs}\delta^3(\mathbf{p}-\mathbf{k})a_{\mathbf{k}}^{r\,\dagger} = E_{\mathbf{p}}a_{\mathbf{p}}^{s\,\dagger}, \tag{20}$$

where we used the result in (15) applying the replacement $b_{\mathbf{p}}^{s\,\dagger} \to a_{\mathbf{p}}^{s\,\dagger}$ and accounting for a minus sign from the order in the commutator. Thus

$$Ha_{\mathbf{p}}^{s\,\dagger}|\psi\rangle = (E_\psi + E_{\mathbf{p}})\,a_{\mathbf{p}}^{s\,\dagger}|\psi\rangle\,. \tag{21}$$

In the same way one can show that

$$Hb_{\mathbf{p}}^{s\,\dagger}|\psi\rangle = (E_\psi + E_{\mathbf{p}})\,b_{\mathbf{p}}^{s\,\dagger}|\psi\rangle\,. \tag{22}$$

These two identities imply that an eigenstate of the Hamiltonian remains as such under the action of the creation operators $a^{s\,\dagger}_{\mathbf{p}}$ and $b^{s\,\dagger}_{\mathbf{p}}$, with its eigenvalue increased by $E_{\mathbf{p}}$. In particular, if we take the vacuum $|0\rangle$ to be state of zero energy (normal-ordered Hamiltonian) we conclude that both $a^{s\,\dagger}_{\mathbf{p}}$ and $b^{s\,\dagger}_{\mathbf{p}}$ create states of energy $E_{\mathbf{p}}$, i.e.

$$H\,|\mathbf{p},s\rangle = H\sqrt{2E_{\mathbf{p}}}a^{s\,\dagger}_{\mathbf{p}}\,|0\rangle = E_{\mathbf{p}}\,|\mathbf{p},s\rangle\,, \tag{23}$$

$$H\,|\overline{\mathbf{p},s}\rangle = H\sqrt{2E_{\mathbf{p}}}b^{s\,\dagger}_{\mathbf{p}}\,|0\rangle = E_{\mathbf{p}}\,|\overline{\mathbf{p},s}\rangle\,. \tag{24}$$

More generally we have

$$\begin{aligned}
H\prod_{\ell=1}^{n} a^{s_{n-\ell+1}\,\dagger}_{\mathbf{p}_{n-\ell+1}} &= H a^{s_n\,\dagger}_{\mathbf{p}_n}\prod_{\ell=1}^{n-1} a^{s_{n-\ell+1}\,\dagger}_{\mathbf{p}_{n-\ell+1}} = a^{s_n\,\dagger}_{\mathbf{p}_n} H\prod_{\ell=1}^{n-1} a^{s_{n-\ell+1}\,\dagger}_{\mathbf{p}_{n-\ell+1}} + \left[H, a^{s_n\,\dagger}_{\mathbf{p}_n}\right]\prod_{\ell=1}^{n-1} a^{s_{n-\ell+1}\,\dagger}_{\mathbf{p}_{n-\ell+1}} = \\
&= a^{s_n\,\dagger}_{\mathbf{p}_n} H\prod_{\ell=1}^{n-1} a^{s_{n-\ell+1}\,\dagger}_{\mathbf{p}_{n-\ell+1}} + E_{\mathbf{p}_n} a^{s_n\,\dagger}_{\mathbf{p}_n}\prod_{\ell=1}^{n-1} a^{s_{n-\ell+1}\,\dagger}_{\mathbf{p}_{n-\ell+1}} = \dots \\
&= \left(\prod_{\ell=1}^{n} a^{s_{n-\ell+1}\,\dagger}_{\mathbf{p}_{n-\ell+1}}\right)\left(H + \sum_{j=1}^{n} E_{\mathbf{p}_j}\right),
\end{aligned} \tag{25}$$

and the same holds for creation operators of antifermions. A state with n fermions and m antifermions can be expressed as

$$|\{\{\mathbf{p}_i, s_i\}\,, i=1,\dots,n\}\,, \left\{\overline{\{\mathbf{p}_i, s_i\}}, i=1,\dots,n+m\right\}\rangle = \prod_{\ell=1}^{n}\sqrt{2E_{\mathbf{p}_\ell}}a^{s_\ell\,\dagger}_{\mathbf{p}_\ell}\prod_{j=1}^{m}\sqrt{2E_{\mathbf{p}_j}}b^{s_j\,\dagger}_{\mathbf{p}_j}\,|0\rangle\,, \tag{26}$$

so we can use (25) to show that this is an eigenstate of the Hamiltonian with eigenvalue

$$\sum_{j=1}^{n+m} E_{\mathbf{p}_j}\,, \tag{27}$$

hence any state with a given number of fermions and antifermions is an eigenstate of the Hamiltonian.

c)

$$\begin{aligned}
\langle 0|\,\psi(x) &= \langle 0|\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}_1}}}\sum_{r_1=\pm}\left(a^{r_1}_{\mathbf{k}_1}e^{-ik_1\cdot x}u^{r_1}(k_1) + b^{r_1\,\dagger}_{\mathbf{k}_1}e^{ik_1\cdot x}v^{r_1}(k_1)\right) \\
&= \langle 0|\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}_1}}}\sum_{r_1=\pm}\left(a^{r_1}_{\mathbf{k}_1}e^{-ik_1\cdot x}u^{r_1}(k_1)\right)
\end{aligned} \tag{28}$$

$$\langle 0|\,a^{r_1}_{\mathbf{k}_1}e^{-ik_1\cdot x}\left(a^{r_2}_{\mathbf{k}_2}e^{-ik_2\cdot y}u^{r_2}(k_2) + b^{r_1\,\dagger}_{\mathbf{k}_1}e^{ik_2\cdot y}v^{r_2}(k_2)\right) = \langle 0|\,a^{r_1}_{\mathbf{k}_1}e^{-ik_1\cdot x}a^{r_2}_{\mathbf{k}_2}e^{-ik_2\cdot y}u^{r_2}(k_2) \tag{29}$$

$$\begin{aligned}
&\langle 0|\,\psi(x)\psi(y)\psi(z)\,|\{\mathbf{p}_1, s_1\}\,\{\mathbf{p}_2, s_2\}\,\{\mathbf{p}_3, s_3\}\rangle \\
&= \int\left(\prod_{i=1}^{3}\frac{\mathrm{d}^3\mathbf{k}_i}{(2\pi)^3\sqrt{2E_{\mathbf{k}_i}}}\right)\sum_{r_1,r_2,r_3=\pm} u^{r_1}(k_1)u^{r_2}(k_2)u^{r_3}(k_3)e^{-i(k_1\cdot x+k_2\cdot y+k_3\cdot z)} \\
&\times\sqrt{2E_{\mathbf{p}_1}}\sqrt{2E_{\mathbf{p}_2}}\sqrt{2E_{\mathbf{p}_3}}\,\langle 0|\,a^{r_1}_{\mathbf{k}_1}a^{r_2}_{\mathbf{k}_2}a^{r_3}_{\mathbf{k}_3}a^{s_1\,\dagger}_{\mathbf{p}_1}a^{s_2\,\dagger}_{\mathbf{p}_2}a^{s_3\,\dagger}_{\mathbf{p}_3}\,|0\rangle\,.
\end{aligned} \tag{30}$$

$$a^{r_j}_{\mathbf{k}_j}\prod_{\ell=1}^{3}a^{s_\ell\,\dagger}_{\mathbf{p}_\ell}\,|0\rangle = \sum_{i_1=1}^{3}(-1)^{i_1}(2\pi)^3\delta^3(\mathbf{p}_{i_1}-\mathbf{k}_j)\delta^{r_j s_{i_1}}\prod_{\substack{\ell=1\\ \ell\neq i_1}}^{3}a^{s_\ell\,\dagger}_{\mathbf{p}_\ell}\,|0\rangle \tag{31}$$

$$\prod_{j=1}^{3} a^{r_j}_{\mathbf{p}_j} \prod_{\ell=1}^{3} a^{s_\ell\,\dagger}_{\mathbf{p}_\ell} |0\rangle = (2\pi)^9 \sum_{i_1=1}^{3} \sum_{\substack{i_2=1\\ i_2\neq i_1}}^{3} \sum_{\substack{i_3=1\\ i_3\neq i_1,i_2}}^{3} (-1)^{i_1+i_2+i_3}\, \delta^{r_1 s_{i_1}} \delta^{r_2 s_{i_2}} \delta^{r_3 s_{i_3}}$$

$$\delta^3(\mathbf{p}_{i_1} - \mathbf{k}_1)\, \delta^3(\mathbf{p}_{i_2} - \mathbf{k}_2)\, \delta^3(\mathbf{p}_{i_2} - \mathbf{k}_3)\, |0\rangle \ , \tag{32}$$

which can be expressed in a much compact form realising that the sum is just a sum over all permutations, including the sign of the actual permutation, i.e.

$$\prod_{j=1}^{3} a^{r_j}_{\mathbf{k}_j} \prod_{\ell=1}^{3} a^{s_\ell\,\dagger}_{\mathbf{p}_\ell} |0\rangle = (2\pi)^9 \sum_{\sigma\in\mathcal{S}_3} \mathrm{sgn}(\sigma) \prod_{i=1}^{3} \delta^{r_i s_{\sigma(i)}} \delta^3(\mathbf{p}_{\sigma(i)} - \mathbf{k}_i)\,. \tag{33}$$

If we plug (33) in (30), after relabelling $x \to x_1$, $y \to x_2$, $z \to x_3$, integrating over the momenta $\mathbf{k}_i$ and summing over spin configurations r_i we get

$$\langle 0|\, \psi(x_1)\psi(x_2)\psi(x_3)\, |\{\mathbf{p}_1, s_1\}\,\{\mathbf{p}_2, s_2\}\,\{\mathbf{p}_3, s_3\}\rangle = \sum_{\sigma\in\mathcal{S}_3} \mathrm{sgn}(\sigma) \prod_{i=1}^{3} u^{s_{\sigma(i)}}(p_{\sigma(i)}) e^{-ip_{\sigma(i)}\cdot x_i}\,. \tag{34}$$

Let us define the matrix F as

$$F_{mn} = u^{s_n}(p_n) e^{-ip_n\cdot x_m}, \qquad \text{with} \qquad m,n = 1,2,3\,. \tag{35}$$

It follows that (34) is simply the determinant of the matrix F, i.e.

$$\langle 0|\, \psi(x_1)\psi(x_2)\psi(x_3)\, |\{\mathbf{p}_1, s_1\}\,\{\mathbf{p}_2, s_2\}\,\{\mathbf{p}_3, s_3\}\rangle = \det(F)\,. \tag{36}$$

3. Noether's Theorem I

Consider the transformation of the field $\phi(x)$ as

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x), \tag{37}$$

which leaves the equations of motion invariant. The Lagrangian density thus must be invariant up to a to 4-divergence term:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \partial_\mu \mathcal{J}^\mu(x)\,. \tag{38}$$

Noether's theorem states

$$\partial_\mu j^\mu(x) = 0 \tag{39}$$

where the conserved current $j^\mu(x)$ is defined as

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - \mathcal{J}^\mu\,. \tag{40}$$

In the case of the free Lagrangian for a complex Klein-Gordon field we have a global phase symmetry

$$\phi(x) \to \phi'(x) = e^{i\theta}\phi(x), \qquad \phi^\dagger(x) \to \phi'^\dagger(x) = \phi^\dagger(x)e^{-i\theta}\,. \tag{41}$$

The Lagrangian density itself is invariant under this transformation, hence there is no surface term involved and the Noether's current reads

$$j^\mu_{\text{free}}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \delta\phi^\dagger \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\,. \tag{42}$$

Considering an infinitesimal transformation $\phi(x) \to \phi(x) + \delta\phi(x) = \phi(x) + i\theta\phi(x) + \mathcal{O}(\theta^2)$, and the equivalent for $\phi^\dagger(x)$ we get

$$j^\mu_{\text{free}}(x) = i\left[\left(\partial^\mu \phi^\dagger\right)\phi(x) - \phi^\dagger(x)\partial^\mu\phi(x)\right]\,. \tag{43}$$

a) If we rewrite the Lagrangian density as

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}\,, \tag{44}$$

and notice that the product $\phi^\dagger\phi$ is manifestly invariant under a phase transformation we get $\mathcal{J}^\mu = 0$ and so that the Noether's current reads

$$j^\mu(x) = j^\mu_{\text{free}}(x) + \frac{\partial \mathcal{L}_{\text{int}}}{\partial(\partial_\mu \phi)}\delta\phi + \delta\phi^\dagger \frac{\partial \mathcal{L}_{\text{int}}}{\partial(\partial_\mu \phi^\dagger)}\,. \tag{45}$$

The interaction term does not depend on the field derivatives, thus the current is the same as in the free case.

b) Also in this case $J^\mu = 0$ since the Lagrangian is manifestly invariant under a phase transformation of the fields, however the interaction term depends on field derivatives. Explicitly:

$$\mathcal{L}_{\text{int}} = \lambda g^{\mu\nu}\left[(\partial_\mu\phi^\dagger)\phi\,(\partial_\nu\phi^\dagger)\phi + \phi^\dagger\partial_\mu\phi\,(\partial_\nu\phi^\dagger)\phi + (\partial_\mu\phi^\dagger)\phi\,\phi^\dagger\partial_\nu\phi + \phi^\dagger(\partial_\mu\phi)\,\phi^\dagger\partial_\nu\phi\right]\,, \tag{46}$$

therefore we get

$$\begin{aligned}
\frac{\partial \mathcal{L}_{\text{int}}}{\partial(\partial_\mu\phi)} &= \lambda\left[\phi^\dagger(\partial^\mu\phi^\dagger)\phi + (\partial^\mu\phi^\dagger)\phi\phi^\dagger + \phi^\dagger\phi^\dagger(\partial^\mu\phi) + \phi^\dagger(\partial^\mu\phi)\phi^\dagger\right] \\
&= \lambda\left[\phi^\dagger\partial^\mu(\phi^\dagger\phi) + \partial^\mu(\phi^\dagger\phi)\phi^\dagger\right]\,, \qquad (47) \\
\frac{\partial \mathcal{L}_{\text{int}}}{\partial(\partial_\mu\phi^\dagger)} &= \lambda\left[\phi\,(\partial^\mu\phi^\dagger)\phi + (\partial^\mu\phi^\dagger)\phi\phi + \phi^\dagger(\partial^\mu\phi)\phi + \phi\phi^\dagger(\partial^\mu\phi)\right] \\
&= \lambda\left[\phi\,\partial^\mu(\phi^\dagger\phi) + \partial^\mu(\phi^\dagger\phi)\phi\right] \qquad (48)
\end{aligned}$$

and for the field variations $\delta\phi(x) = i\phi(x)$ and $\delta\phi^\dagger(x) = -i\phi^\dagger(x)$. Putting everything together, the conserved current in this case reads

$$\begin{aligned} j^\mu(x) &= j^\mu_{\text{free}}(x) + i\lambda \left[\phi^\dagger \partial^\mu(\phi^\dagger\phi)\phi + \partial^\mu(\phi^\dagger\phi)(\phi^\dagger\phi) - (\phi^\dagger\phi)\, \partial^\mu(\phi^\dagger\phi) - \phi^\dagger\partial^\mu(\phi^\dagger\phi)\phi\right] \\ &= j^\mu_{\text{free}}(x) + i\lambda \left[\partial^\mu(\phi^\dagger\phi)(\phi^\dagger\phi) - (\phi^\dagger\phi)\, \partial^\mu(\phi^\dagger\phi)\right] . \end{aligned} \tag{49}$$

c) The interaction Lagrangian is not manifestly invariant under a global phase transformation of the fields, and moreover an infinitesimal transformation of the fields $\phi \to \phi + i\theta\phi$ does not change the Lagrangian by simply a surface term, i.e.

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}, \tag{50}$$

where now $\delta\mathcal{L}$ is a polynomial in the fields ϕ and $\phi^\dagger$. The Lagrangian is invariant under the transformation $\phi \to -\phi$, which, on the other hand, is not continuous and therefore, there is no associated Noether's current.

d) The Dirac Lagrangian is invariant under a global phase transformation of the fields

$$\psi(x) \to \psi'(x) = e^{i\theta}\psi(x), \qquad \psi^\dagger(x) \to \psi'^\dagger(x) = \psi^\dagger(x)e^{-i\theta}, \tag{51}$$

therefore we have the conserved current

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \delta\psi^\dagger \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)} . \tag{52}$$

However the Lagrangian does not depend explicitly on $\partial_\mu\psi^\dagger$, thus the current simply reads

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = i\bar{\psi}\gamma^\mu\psi\,, \tag{53}$$

where we used $\delta\psi = i\psi$.

4. Cancellation of vacuum bubble diagrams

Let us focus first on the numerator of the vacuum expectation value G_K

$$\tilde{G}_K \equiv \lim_{T\to\infty(1-i\epsilon)} \langle 0|\,\mathcal{T}\left\{\prod_{i=1}^{2K}\phi_i \exp\left(-i\int_{-T}^{T}\mathrm{d}t\int \mathrm{d}^3\mathbf{y}\frac{\lambda}{4!}\phi^4(y)\right)\right\}|0\rangle\ . \tag{54}$$

If we introduce the shorthand notation

$$\left[-i\int H(\phi_j)\right] \equiv -i\int \mathrm{d}y_j\,\frac{\lambda}{4!}\phi^4(y_j), \tag{55}$$

the expansion in powers of λ, i.e. in power of the Hamiltonian, reads

$$\tilde{G}_K \equiv \langle 0|\,\mathcal{T}\left\{\prod_{i=1}^{2K}\phi_i\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}\left[-i\int H(\phi_j)\right]\right\}|0\rangle\ . \tag{56}$$

According to Wick's theorem, for a fixed order in n, only the fully contracted diagrams with $2K$ external points and n internal ones contribute. Therefore, for given n we will have a sum of diagrams which contain either connected graphs only (i.e. no vacuum graphs) or connected and disconnected subgraphs. Thus

$$\begin{aligned}&\langle 0|\,\mathcal{T}\left\{\prod_{i=1}^{2K}\phi_i\prod_{j=1}^{n}\left[-i\int H(\phi_j)\right]\right\}|0\rangle\\ &=\sum_{m=0}^{n}\binom{n}{m}\langle 0|\,\mathcal{T}\left\{\prod_{i=1}^{2K}\phi_i\prod_{j=1}^{m}\left[-i\int H(\phi_j)\right]\right\}|0\rangle_c\,\langle 0|\,\mathcal{T}\left\{\prod_{j=m+1}^{n}\left[-i\int H(\phi_j)\right]\right\}|0\rangle\ ,\end{aligned} \tag{57}$$

where with the subscript c we denote the connected subgraphs. The combinatorial factor comes from the fact that there are $\binom{n}{m}$ ways to choose m internal points out of n to fully contract with the external fields. If we now sum over all n, the numerator takes the form

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{m!}\langle 0|\,\mathcal{T}\left\{\prod_{i=1}^{2K}\phi_i\prod_{j=1}^{m}\left[-i\int H(\phi_j)\right]\right\}|0\rangle_c\,\langle 0|\,\mathcal{T}\left\{\frac{1}{(n-m)!}\prod_{j=m+1}^{n}\left[-i\int H(\phi_j)\right]\right\}|0\rangle\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{m!}\langle 0|\,\mathcal{T}\left\{\prod_{i=1}^{2K}\phi_i\prod_{j=1}^{m}\left[-i\int H(\phi_j)\right]\right\}|0\rangle_c\,\langle 0|\,\mathcal{T}\left\{\frac{1}{(n-m)!}\prod_{j=1}^{n-m}\left[-i\int H(\phi_j)\right]\right\}|0\rangle\end{aligned} \tag{58}$$

which can be factorised as

$$\left(\sum_{m=0}^{\infty}\frac{1}{m!}\langle 0|\,\mathcal{T}\left\{\prod_{i=1}^{2K}\phi_i\prod_{j=1}^{m}\left[-i\int H(\phi_j)\right]\right\}|0\rangle_c\right)\sum_{n=0}^{\infty}\langle 0|\,\mathcal{T}\left\{\frac{1}{n!}\prod_{j=1}^{n}\left[-i\int H(\phi_j)\right]\right\}|0\rangle\ . \tag{59}$$

The last term on the right hand side exponentiates

$$\left(\sum_{m=0}^{\infty}\frac{1}{m!}\langle 0|\,\mathcal{T}\left\{\prod_{i=1}^{2K}\phi_i\prod_{j=1}^{m}\left[-i\int H(\phi_j)\right]\right\}|0\rangle_c\right)\langle 0|\exp\left(-i\int \mathrm{d}t\,H(\phi(y))\right)|0\rangle \tag{60}$$

and this exactly cancels the denominator in the vacuum expectation value. Hence, if we expand to any $\mathcal{O}(\lambda^L)$ we can discard all diagrams containing vacuum subgraphs.

5. Feynman Diagrams

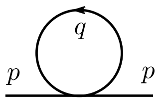
We are interested in the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\,. \tag{61}$$

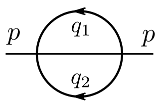
In this exercise we will locate external fields at points x_i, and interaction points are defined at y_i. For example, for four-point diagrams, external fields $\phi(x_i)$, $i = 1, \ldots, 4$ will be labelled as $1, 2, 3, 4$.
For the ϕ^4 theory, there is a diagrammatic recipe to find the multiplicity factor of a Feynman diagram.

1. Draw a dot for each external point x_i and a vertex with four lines coming out of it for each internal point y_i.
2. Start by connecting all the external lines to internal lines to form the diagram in question. each time you connect a line to a vertex count how many ways you could have done this, generating the same diagram.
3. Now form all the loops by picking an unconnected line from the vertex and keeping track of how many ways it can be connected generating the same diagram - take care not to double count.
4. For a diagram with N internal points (vertices) the multiplicity factor is given by
$$\frac{1}{N!}\frac{1}{(4!)^N} \tag{62}$$
times the product of the number calculated above. Here the $1/4!$ factor comes from the interaction term in the Lagrangian.

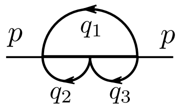
a) For the two point diagrams listed we have:



$$= \frac{(-i\lambda)}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{q^2 - m^2}, \tag{63}$$

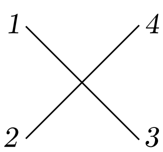


$$= \frac{(-i\lambda)^2}{6}\int \frac{\mathrm{d}^4 q_1}{(2\pi)^4}\frac{\mathrm{d}^4 q_2}{(2\pi)^4}\frac{1}{(q_1^2 - m^2)(q_2^2 - m^2)((p+q_1+q_2)^2 - m^2)}, \tag{64}$$



$$= \frac{(-i\lambda)^3}{12}\int \frac{\mathrm{d}^4 q_1}{(2\pi)^4}\frac{\mathrm{d}^4 q_2}{(2\pi)^4}\frac{\mathrm{d}^4 q_3}{(2\pi)^4}\left[\frac{1}{(q_1^2 - m^2)(q_2^2 - m^2)(q_3^2 - m^2)((p+q_1+q_2)^2 - m^2)((p+q_1+q_3)^2 - m^2)}\right]. \tag{65}$$

b) At $\mathcal{O}(\lambda)$ we have just one tree-level diagram.



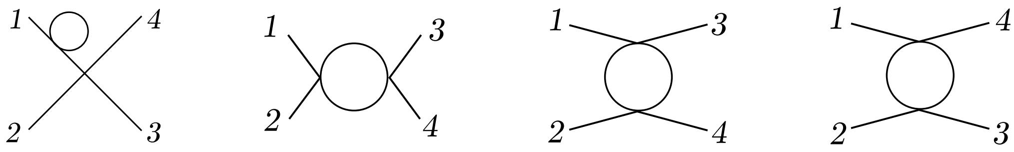
d1.1

There are 4! possible ways to connect an internal point to the 4 external ones. If we include the $1/4!$ coupling from the Lagrangian, the multiplicity factor reads

$$S(\mathrm{d}1.1) = 4!\frac{1}{4!} = 1\,. \tag{66}$$

therefore the 1/4! normalisation in the Lagrangian ensures that the tree-level Feynman rule is simply $(-i\lambda)$.

At $\mathcal{O}(\lambda^2)$ we are dealing with one-loop diagrams. There are 7 independent diagrams:



d2.1 d2.2 d2.3 d2.4

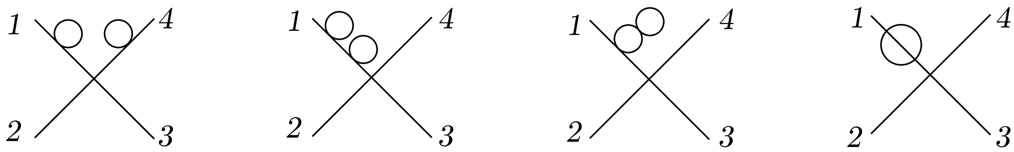
4 of type (d2.1) where the one-loop insertion is on each external leg and 3 diagrams with a one-loop insertion between external legs. Diagram (d2.2) is referred to as s-channel, (d2.3) as t-channel and (d2.4) as u-channel. The multiplicity factors are:

$$S(\mathrm{d}2.1) = \frac{1}{2!}\frac{1}{(4!)^2} 2\cdot 4\cdot 4\cdot 3\cdot 2\cdot 3 = \frac{1}{2}, \tag{67}$$

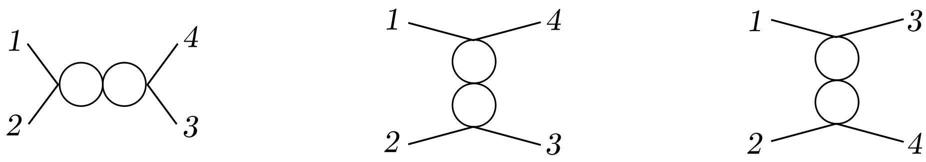
$$S(\mathrm{d}2.2-4) = \frac{1}{2!}\frac{1}{(4!)^2} 2\cdot 4\cdot 3\cdot 4\cdot 3\cdot 2 = \frac{1}{2}\,. \tag{68}$$

Diagram in (d2.1) is not irreducible, therefore its multiplicity factor is the product of its irreducible subdiagrams, as one can check from (63) and (66).

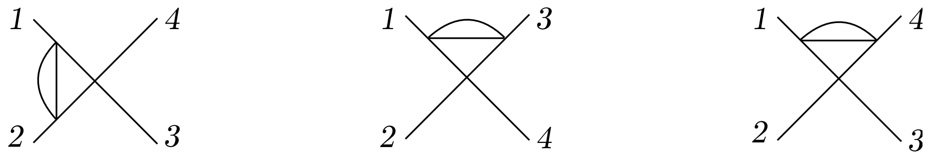
At $\mathcal{O}(\lambda^3)$ we have to consider two-loop diagrams. There are 42 independent diagrams:



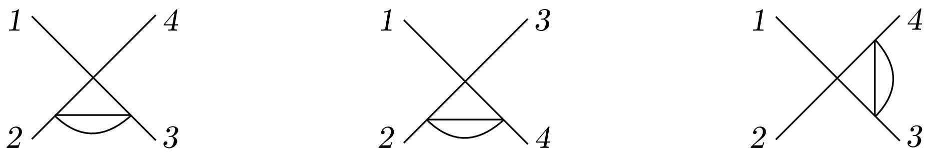
d3.1 d3.2 d3.3 d3.4



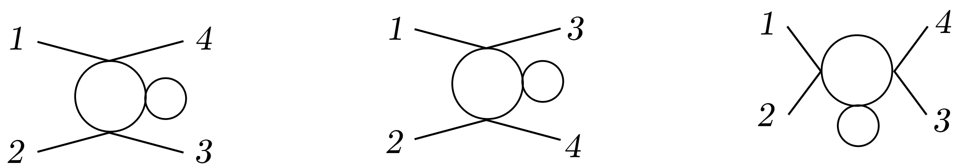
d3.5 d3.6 d3.7



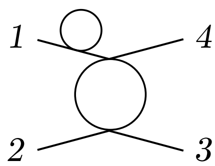
d3.8 d3.9 d3.10



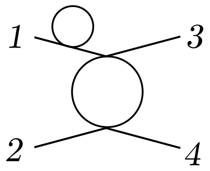
d3.11 d3.12 d3.13



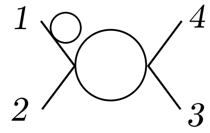
d3.14 d3.15 d3.16



d3.17



d3.18



d3.19

6 of type (d3.1) where we have a one-loop insertion on two different external legs; 4 of type (d3.2) where we have two reducible one-loop insertions on the same external leg; 4 of type (d3.3) where the loop correction is placed on each external leg; 4 of type (d3.4) where the two-loop irreducible correction is on each external leg; diagrams (d3.5-16) are irreducible and each is counted once; for (d3.17-19) we can have one-loop insertion on each external leg of the s, t and u-channel diagrams, therefore 4 each. The multiplicity factors read:

$$S(\text{d3.1}) = \frac{1}{3!}\frac{1}{(4!)^3} 3\cdot 2\cdot 4\cdot 4\cdot 4\cdot 3\cdot 2\cdot 3\cdot 3 = \frac{1}{4}, \tag{69}$$

$$S(\text{d3.2}) = \frac{1}{3!}\frac{1}{(4!)^3} 3\cdot 2\cdot 4\cdot 4\cdot 3\cdot 2\cdot 4\cdot 3\cdot 3 = \frac{1}{4}, \tag{70}$$

$$S(\text{d3.3}) = \frac{1}{3!}\frac{1}{(4!)^3} 3\cdot 2\cdot 4\cdot 4\cdot 3\cdot 2\cdot 3\cdot 4\cdot 3 = \frac{1}{4}, \tag{71}$$

$$S(\text{d3.4}) = \frac{1}{3!}\frac{1}{(4!)^3} 3\cdot 2\cdot 4\cdot 4\cdot 3\cdot 2\cdot 4\cdot 3\cdot 2 = \frac{1}{6}, \tag{72}$$

$$S(\text{d3.5}-7) = \frac{1}{3!}\frac{1}{(4!)^3} 3\cdot 2\cdot 4\cdot 3\cdot 4\cdot 3\cdot 4\cdot 3\cdot 2 = \frac{1}{4}, \tag{73}$$

$$S(\text{d3.8}-13) = \frac{1}{3!}\frac{1}{(4!)^3} 3\cdot 2\cdot 4\cdot 3\cdot 4\cdot 4\cdot 2\cdot 3\cdot 3\cdot 2 = \frac{1}{2}, \tag{74}$$

$$S(\text{d3.14}-16) = \frac{1}{3!}\frac{1}{(4!)^3} 3\cdot 2\cdot 4\cdot 3\cdot 4\cdot 3\cdot 4\cdot 3\cdot 2\cdot 2 = \frac{1}{2}, \tag{75}$$

$$S(\text{d3.17}-19) = \frac{1}{3!}\frac{1}{(4!)^3} 3\cdot 2\cdot 4\cdot 4\cdot 4\cdot 3\cdot 3\cdot 2\cdot 3 = \frac{1}{4}\,. \tag{76}$$

One can check again that for reducible diagrams the factor is given by the product of its irreducible subdiagrams.

c) In this case $\phi^\dagger(x)$ and $\phi(x)$ are distinct fields (different charge) hence they are distinguishable. Also, in the Lagrangian we now have a factor $\lambda/4$, so for N interaction vertices we need to consider a global factor

$$\frac{1}{N!}\frac{1}{4^N}\,. \tag{77}$$

We can repeat the exercise in a) and b) by considering the fields $\phi^\dagger(x_1)$, $\phi(x_2)$, $\phi^\dagger(x_3)$, $\phi(x_4)$ and label them $1, 2, 3, 4$ respectively. Let us consider first the 4-point fully connected $\mathcal{O}(\lambda)$ diagram. Take the four external fields and try to connect them through an internal point y_1, where there will be two fields $\phi^\dagger$ and two fields ϕ. Chosen $\phi^\dagger(x_1)$, we have two ways to connect it to the internal point y_1, and two ways to connect $\phi(x_2)$ to y_1. The rest is fixed. Thus, we have in total a factor

$$\frac{1}{4}\cdot 2\cdot 2 = 1\,. \tag{78}$$

The factor $1/4$ in the Lagrangian ensures again that the tree level Feynman rule is simply λ. As a consequence, for the first diagram of part b) we get the same answer. For what regards part a) we can consider for definiteness the two point function of a field $\psi^\dagger$. Taken one of the two external points, there are two ways to connect it to the internal vertex and the rest is then fixed, so we have a factor $(1/4)\cdot 2$. Summarising the results for part a), the symmetry factor S_n for the n-loop two-point functions for a complex scalar field are

$$S_1 = \frac{(-i\lambda)}{2}, \qquad S_2 = \frac{(-i\lambda)^2}{2}, \qquad S_3 = \frac{(-i\lambda)^3}{2}. \tag{79}$$

For part b) we have

$$S(\mathrm{d}1.1) = 1, \tag{80}$$
$$S(\mathrm{d}2.1) = \frac{1}{2}, \tag{81}$$
$$S(\mathrm{d}2.2-4) = 1, \tag{82}$$
$$S(\mathrm{d}3.1) = \frac{1}{4}, \tag{83}$$
$$S(\mathrm{d}3.2) = \frac{1}{4}, \tag{84}$$
$$S(\mathrm{d}3.3) = \frac{1}{4}, \tag{85}$$
$$S(\mathrm{d}3.4) = \frac{1}{2}, \tag{86}$$
$$S(\mathrm{d}3.5-7) = \frac{1}{3!}\frac{1}{(4)^3}3\cdot 2^7 = 1, \tag{87}$$
$$S(\mathrm{d}3.8-13) = \frac{1}{3!}\frac{1}{(4)^3}3\cdot 2^7 = 1, \tag{88}$$
$$S(\mathrm{d}3.14-16) = \frac{1}{3!}\frac{1}{(4)^3}3\cdot 2^7 = 1, \tag{89}$$
$$S(\mathrm{d}3.17-19) = \frac{1}{2}\,. \tag{90}$$

We have shown that in this model, up to two loops, irreducible four-point diagrams have a symmetry factor equal to 1. This holds true for any higher-order loop correction.