# QFT Problem Set 4

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**Problem 1.** *Gamma matrix manipulation*

a) $2\not{p}\not{p} = 2p_\mu p^\mu \mathbb{I} \Rightarrow \not{p}\not{p} = p^2\mathbb{I}$

b) $\mathrm{Tr}\not{p}\not{q} = p_\mu q^\mu \mathrm{Tr}\mathbb{I}_4 \Rightarrow \mathrm{Tr}\not{p}\not{q} = 4p.q$

c) $\gamma^5\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = \gamma^2\gamma^3\gamma^2\gamma^3 = \gamma^3\gamma^3 = -1$
$\{\gamma^5, \gamma^0\} = \{\gamma^5, \gamma^1\} = \{\gamma^5, \gamma^2\} = \{\gamma^5, \gamma^3\} = 0 \Rightarrow \{\gamma^5, \gamma^\mu\} = 0$

d) For any odd number of gamma matrices

$$\begin{aligned} &\mathrm{Tr}\{\gamma^\alpha\gamma^\beta\gamma^\delta \dots\} = -\mathrm{Tr}\{\gamma^5\gamma^5\gamma^\alpha\gamma^\beta\gamma^\delta \dots\} \overset{\text{cyclic}}{=} -\mathrm{Tr}\{\gamma^5\gamma^\alpha\gamma^\beta\gamma^\delta \dots \gamma^5\} \\ &\overset{\text{anticommute}}{=} -(-1)^n \mathrm{Tr}\{\gamma^5\gamma^5\gamma^\alpha\gamma^\beta\gamma^\delta \dots\} = -\mathrm{Tr}\{\gamma^\alpha\gamma^\beta\gamma^\delta \dots\}\ , \end{aligned} \tag{1}$$

where $n$ is the number of gamma matrices $\gamma^5$ was anticommuted through. The trace is equal to minus itself hence 0.

e)

$$\mathrm{Tr}\{\not{p}\not{q}\not{r}\not{s}\} = \mathrm{Tr}\{\not{p}\not{q} r_\mu s_\nu(-\gamma^\nu\gamma^\mu + 2g^{\mu\nu})\} = \mathrm{Tr}\{\not{p}\not{q}(2s.r - \not{s}\not{r})\} \tag{2}$$

$$\begin{aligned} \mathrm{Tr}\{\not{p}\not{q}\not{r}\not{s}\} &= \mathrm{Tr}\{p_\alpha q_\beta r_\mu s_\nu(-\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu + 2g^{\mu\nu}\gamma^\alpha\gamma^\beta - 2g^{\beta\nu}\gamma^\alpha\gamma^\mu + 2g^{\alpha\nu}\gamma^\beta\gamma^\mu)\} \\ 2\mathrm{Tr}\{\not{p}\not{q}\not{r}\not{s}\} &= 2\mathrm{Tr}\{\not{p}\not{q}s.r - \not{p}\not{r}s.q + \not{q}\not{r}s.p\}\text{Use b)} \\ \mathrm{Tr}\{\not{p}\not{q}\not{r}\not{s}\} &= 4\,(p.qs.r - p.rq.s + p.sq.r) \end{aligned} \tag{3}$$

**Problem 2.** *Scalar-fermion scattering*

Diagrams to consider:

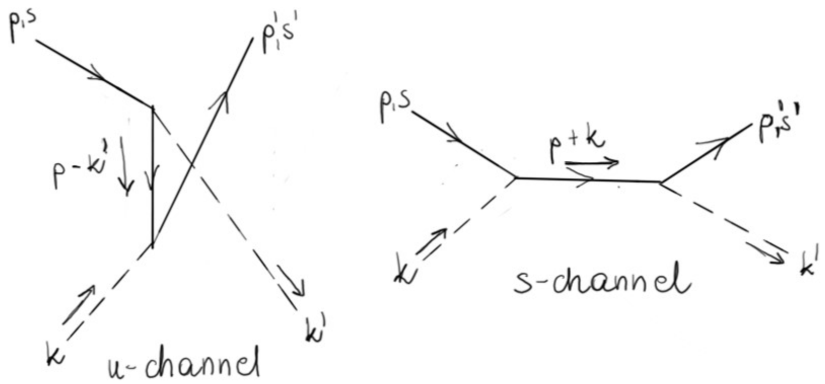


Figure 1: Fermion-scalar scattering.

a) Useful relations for Dirac spinors:

$$
\begin{aligned}
(\not{p} - m)u_s(p) &= 0 \\
\bar{u}_s(p)(\not{p} - m) &= 0
\end{aligned}
\tag{4}
$$

Define $S = (p+k)^2$, $T = (p-p')^2$ and $U = (p-k')^2$. In the following the spinor indices are suppressed. S-channel diagram:

$$
\tilde{M}_1 = -ig^2\bar{u}^{s'}(p')\frac{(\not{p}' + \not{k}' + m)}{S - m^2}u^s(p) = -ig^2\bar{u}^{s'}(p')\frac{(\not{k}' + 2m)}{S - m^2}u^s(p) \tag{5}
$$

U-channel diagram:

$$
\tilde{M}_2 = -ig^2\bar{u}^{s'}(p')\frac{(\not{p} - \not{k}' + m)}{U - m^2}u^s(p) = -ig^2\bar{u}^{s'}(p')\frac{(-\not{k}' + 2m)}{U - m^2}u^s(p) \tag{6}
$$

The scattering matrix element is $\tilde{M} = \tilde{M}_1 + \tilde{M}_2$.

b) Use the results

$$
\sum_s u^s(p)\bar{u}^s(p) = (\not{p} + m) \tag{7}
$$

and

$$
(\bar{u}(p)\gamma^\mu u(k))^\dagger = \bar{u}(k)\gamma^\mu u(p)\ . \tag{8}
$$

The quantity of interest is

$$
\mathcal{T} = \sum_{s,s'}|\tilde{M}|^2 = \sum_{s,s'}|\tilde{M}_1|^2 + |\tilde{M}_2|^2 + \tilde{M}_1^*\tilde{M}_2 + \tilde{M}_1\tilde{M}_2^*\ . \tag{9}
$$

Calculate term by term

$$
\begin{aligned}
\sum_{s,s'}|\tilde{M}_1|^2 &= \frac{g^4}{(S-m^2)^2}\sum_{s,s'}\mathrm{Tr}\{u^{s'}(p')\bar{u}^{s'}(p')(\not{k}' + 2m)u^s(p)\bar{u}^s(p)(\not{k}' + 2m)\} \\
&= \frac{g^4}{(S-m^2)^2}\mathrm{Tr}\{(\not{p}' + m)(\not{k}' + 2m)(\not{p} + m)(\not{k}' + 2m)\} = \\
&= \frac{g^4}{(S-m^2)^2}\{8p'.k'p.k' + 12m^2p'.p + 16m^2(p+p').k' + 20m^4\}\ .
\end{aligned}
\tag{10}
$$

Substitute in for the Mandelstam invariants

$$p.k = p'.k' = \frac{S}{2} - m^2 \ , \qquad p.p' = k.k' = m^2 - \frac{T}{2} \ , \tag{11}$$

$$p.k' = p'.k = m^2 - \frac{U}{2} \ , \qquad S + T + U = 4m^2 \ , \tag{12}$$

and the result follows

$$\sum_{s,s'} |\tilde{M}_1|^2 = \frac{g^4}{(S-m^2)^2}\{(S+2m^2)^2 + (U-6m^2)^2 - T(T+6m^2)\} \ . \tag{13}$$

The calculation of the remaining terms is analogous.

c) The final spin direction is not measured, hence we have to sum over all the possibilities. The initial spin is unknown, so we need to average over the possible two spin directions as it can be either.

**Problem 3.** *Fermion-fermion scattering*

Diagrams to consider are:

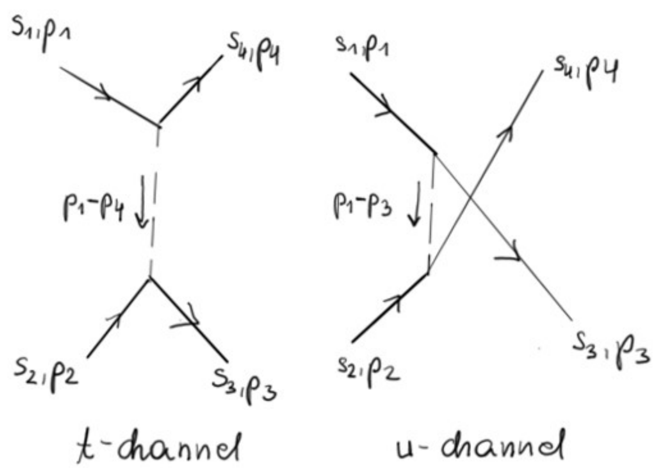


Figure 2: Fermion-fermion scattering.

a) The matrix element is

$$\tilde{M} = -ig^2 \left\{ \frac{\bar{u}^{s_3}(p_3)u^{s_2}(p_2)\bar{u}^{s_4}(p_4)u^{s_1}(p_1)}{T-m^2} - \frac{\bar{u}^{s_3}(p_3)u^{s_1}(p_1)\bar{u}^{s_4}(p_4)u^{s_2}(p_2)}{U-m^2} \right\} \ , \tag{14}$$

where $T = (p_2 - p_3)^2$ and $U = (p_2 - p_4)^2$ and spinor indices are suppressed.

b)

$$\mathcal{T} = \sum_{s_1,s_2,s_3,s_4} |\tilde{M}|^2 = \sum_{s_1,s_2,s_3,s_4} |\tilde{M}_T|^2 + |\tilde{M}_U|^2 + \tilde{M}_T^* \tilde{M}_U + \tilde{M}_T \tilde{M}_U^* \tag{15}$$

Calculate term by term

$$\sum_{s_1,s_2,s_3,s_4} |\tilde{M}_T|^2 = \frac{g^4}{(T-m^2)^2} \sum_{s_1,s_2,s_3,s_4} \mathrm{Tr}\{u^{s_3}(p_3)\bar{u}^{s_3}(p_3)u^{s_2}(p_2)\bar{u}^{s_2}(p_2)\} \times \mathrm{Tr}\{u^{s_4}(p_4)\bar{u}^{s_4}(p_4)u^{s_1}(p_2)\bar{u}^{s_1}(p_1)\}$$
$$= \frac{g^4}{(T-m^2)^2} \mathrm{Tr}\{(\not{p}_3+m)(\not{p}_2+m)\}\mathrm{Tr}\{(\not{p}_4+m)(\not{p}_1+m)\}$$
$$= \frac{16g^4}{(T-m^2)^2}(p_3.p_2+m^2)(p_4.p_1+m^2) \ . \tag{16}$$

Analogously

$$\sum_{s_1,s_2,s_3,s_4} |\tilde{M}_U|^2 = \frac{16g^4}{(U-m^2)^2}(p_3.p_1+m^2)(p_4.p_2+m^2) \ . \tag{17}$$

The cross-terms are

$$\sum_{s_1,s_2,s_3,s_4} \tilde{M}_T^*\tilde{M}_U + \tilde{M}_T\tilde{M}_U^* = -\frac{g^4}{(T-m^2)(U-m^2)} \sum_{s_1,s_2,s_3,s_4} \Big(\mathrm{Tr}\{u^{s_3}(p_3)\bar{u}^{s_3}(p_3)u^{s_2}(p_2)\bar{u}^{s_2}(p_2)u^{s_4}(p_4)\bar{u}^{s_4}(p_4)u^{s_1}(p_1)\bar{u}^{s_1}(p_1)\} + \mathrm{Tr}\{u^{s_3}(p_3)\bar{u}^{s_3}(p_3)u^{s_1}(p_1)\bar{u}^{s_1}(p_1)u^{s_4}(p_4)\bar{u}^{s_4}(p_4)u^{s_2}(p_2)\bar{u}^{s_2}(p_2)\}\Big)$$
$$= -\frac{g^4}{(T-m^2)(U-m^2)} \times \Big(\mathrm{Tr}\{(\not{p}_3+m)(\not{p}_2+m)(\not{p}_4+m)(\not{p}_1+m)\} + \mathrm{Tr}\{(\not{p}_3+m)(\not{p}_1+m)(\not{p}_4+m)(\not{p}_2+m)\}\Big)$$
$$= -\frac{8g^4}{(T-m^2)(U-m^2)} \times \{p_3.p_2p_4.p_1 - p_3.p_4p_2.p_1 + p_3.p_1p_4.p_2 + m^2\sum_{i=1}^{4}\sum_{j\neq i} p_i.p_j + m^4\} \ . \tag{18}$$

The Mandelstam invariants are

$$p_1.p_2 = p_3.p_4 = \frac{S}{2} - m^2 \ , \qquad p_2.p_3 = p_1.p_4 = m^2 - \frac{T}{2} \ , \tag{19}$$
$$p_2.p_4 = p_1.p_3 = m^2 - \frac{U}{2} \ , \qquad S+T+U = 4m^2 \ . \tag{20}$$

The final result is

$$\mathcal{T} = \sum_{s_1,s_2,s_3,s_4} |\tilde{M}|^2 = \frac{4g^4(S+U)^2}{(T-m^2)^2} + \frac{4g^4(S+T)^2}{(U-m^2)^2} + \frac{4g^4(TU-4m^2S)}{(T-m^2)(U-m^2)} \ . \tag{21}$$

**Problem 4.** *One loop*

a) The Feynman diagram

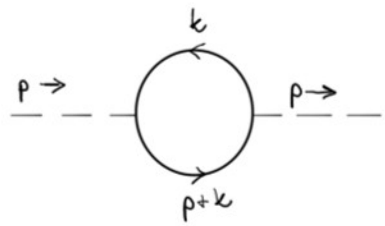


Figure 3: Scalar two-point function contribution.

is described by

$$i\Sigma(p) = -g^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{Tr}\{(\not{p} + \not{k} + m)(\not{k} + m)\}}{[(p+k)^2 - m^2][k^2 - m^2]} , \tag{22}$$

where $m$ has a small imaginary part. Note an additional minus sign associated with a fermion loop. Using the Feynman's parametrisation, the denominator becomes

$$\begin{aligned} \frac{1}{[(p+k)^2 - m^2][k^2 - m^2]} &= \int_0^1 \frac{\mathrm{d}x}{[(1-x)(k^2 - m^2) + x((p+k)^2 - m^2)]^2} \\ &= \int_0^1 \frac{\mathrm{d}x}{(l^2 - M^2)^2} , \end{aligned} \tag{23}$$

where $l = k + px$, $M^2 = m^2 - xp^2(1-x)$ and $M^2$ has a small imaginary component. The numerator is

$$\mathrm{Tr}\{(\not{p} + \not{k} + m)(\not{k} + m)\} = 4(p.p + k^2 + m^2) = 4(l^2 + M^2 + p.l(1-2x)) , \tag{24}$$

where the terms linear in $l$ will vanish due to symmetry. Hence we have

$$\begin{aligned} i\Sigma(p) &= -g^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{4(l^2 + M^2)}{(l^2 - M^2)^2} \\ &= -4g^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \left( \frac{l^2 - M^2}{(l^2 - M^2)^2} + \frac{2M^2}{(l^2 - M^2)^2} \right) \\ &= -4g^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \left( \frac{1}{(l^2 - M^2)} + \frac{2M^2}{(l^2 - M^2)^2} \right) . \end{aligned} \tag{25}$$

Perform Wick's rotation. By standard arguments the temporal part of the integral along the real axis is the same as along the imaginary axis. Replace $l_0 = il_{0,E}$ and $l^2 = -l_E^2$

$$i\Sigma(p) = -4ig^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 l_E}{(2\pi)^4} \left( \frac{1}{(l_E^2 + M^2)} + \frac{2M^2}{(l_E^2 + M^2)^2} \right) \tag{26}$$

and evaluate the Euclidean integrals (done in class). Define

$$I_0 = -i \int \frac{\mathrm{d}^4 l_E}{(2\pi)^4} \frac{1}{(l_E^2 + M^2)} \ , \tag{27}$$

$$I_1 = -i \int \frac{\mathrm{d}^4 l_E}{(2\pi)^4} \frac{1}{(l_E^2 + M^2)^2} \ . \tag{28}$$

After imposing an ultraviolet cut-off $|l_E| < \Lambda$, the results needed are

$$I_0(\Lambda) = -\frac{i}{16\pi^2} \left( \Lambda^2 - M^2 \log \frac{\Lambda^2 + M^2}{M^2} \right) \ , \tag{29}$$

$$I_1(\Lambda) = -\frac{i}{16\pi^2} \left( \log \frac{\Lambda^2 + M^2}{M^2} + \frac{M^2}{\Lambda^2 + M^2} - 1 \right) \ . \tag{30}$$

The propagator correction becomes

$$i\Sigma(p) = -\frac{ig^2}{4\pi^2} \int_0^1 \mathrm{d}x \left( \Lambda^2 + M^2 \log \frac{\Lambda^2 + M^2}{M^2} + \frac{2M^4}{\Lambda^2 + M^2} - 2M^2 \right) \ . \tag{31}$$

Leading order contribution is going to arise from the term

$$\begin{aligned} i\Sigma(p) &= -\frac{ig^2}{4\pi^2} \int_0^1 \mathrm{d}x \left( \Lambda^2 + M^2 \log \frac{\Lambda^2 + M^2}{M^2} \right) \\ &\approx -\frac{ig^2}{4\pi^2} \int_0^1 \mathrm{d}x \left( \Lambda^2 + (m^2 - p^2 x(1-x)) \log \frac{\Lambda^2}{m^2} \right) \\ &\approx \frac{ig^2}{4\pi^2} \left( \frac{1}{6} p^2 \log \frac{\Lambda^2}{m^2} + \Lambda^2 - m^2 \log \frac{\Lambda^2}{m^2} \right) \ . \end{aligned} \tag{32}$$

Mass and momentum renormalisation can be determined by comparing the obtained expression with the formula for the propagator

$$D(p) = \frac{1}{p^2 - m^2 + \Sigma(p) + i\epsilon} \ . \tag{33}$$

The leading order mass shift is $\delta m^2 = \alpha\Lambda^2$ and the kinetic renormalisation contribution is $p^2(1 + \frac{1}{6}\alpha \log \frac{\Lambda^2}{m^2})$, where $\alpha = \frac{g^2}{4\pi^2}$.

b) Feynman diagram

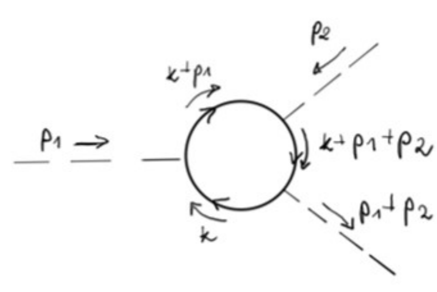


Figure 4: Scalar three-point function contribution.

is described by

$$iV_3 = -g^3 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{Tr}\{(\not{p}_1 + \not{p}_2 + \not{k} + m)(\not{p}_1 + \not{k} + m)(\not{k} + m)\}}{[(p_1 + p_2 + k)^2 - m^2][(p_1 + k)^2 - m^2][k^2 - m^2]} \ . \tag{34}$$

The trace of an odd number of $\gamma$ matrices vanishes, hence the leading momentum power in the numerator is

$$3m\mathrm{Tr}\{\not{k}\not{k}\} = 12mk^2 \ . \tag{35}$$

To keep track of the "worse" divergence consider also only the leading power of momentum in the denominator. Then

$$iV_3 \approx -g^3 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{mk^2}{(k^2 - m^2)^3} \approx -mg^3 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} \to -g^3 m \log \frac{\Lambda}{m^2} + \dots \tag{36}$$

as in the case of the propagator. This divergence can be cancelled by a counter-term proportional to $\phi^3$.

c) Feynman diagram

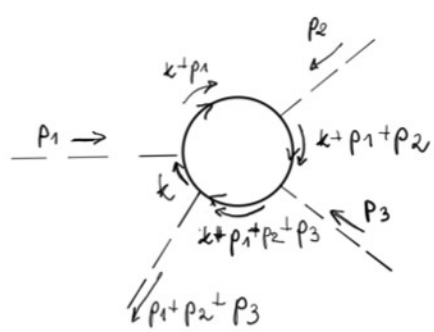


Figure 5: Scalar three-point function contribution.

is described by

$$iV_4 = -g^4 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{Tr}\{(\not{p}_1 + \not{p}_2 + \not{p}_3 + \not{k} + m)(\not{p}_1 + \not{p}_2 + \not{k} + m)(\not{p}_1 + \not{k} + m)(\not{k} + m)\}}{[(p_1 + p_2 + p_3 + k)^2 - m^2][(p_1 + p_2 + k)^2 - m^2][(p_1 + k)^2 - m^2][k^2 - m^2]} \ . \tag{37}$$

Again, consider also only the leading powers of the momentum $k$

$$iV_4 \approx -g^4 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^4}{(k^2 - m^2)^4} \approx -g^4 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} \to -g^4 \log \frac{\Lambda}{m^2} + \dots \ , \tag{38}$$

by exactly the same argument as before. This divergence can be cancelled by a counter-term proportional to $\phi^4$. By simple power counting it can be asserted that any higher order scalar vortex corrections are UV finite.